RESEARCH ARTICLE

WILEY

Solutions for zero-sum two-player games with noncompact decision sets and unbounded payoffs

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Abstract

This article provides sufficient conditions for the existence of solutions for two-person zero-sum games with inf/sup-compact payoff functions and with possibly noncompact decision sets for both players. Payoff functions may be unbounded, and we do not assume any convexity/concavity-type conditions. For such games expected payoff may not exist for some pairs of strategies. The results of this article imply several classic facts. The article also provides sufficient conditions for the existence of a value and solutions for each player. The results of this article are illustrated with the number guessing game.

KEYWORDS

noncompact action sets, solution, two-person game, unbounded payoffs, value

1 | INTRODUCTION

This article studies two-person zero-sum games with possibly noncompact decision sets and unbounded payoff functions, which may not satisfy any convexity or concavity conditions. The payoff functions may be bounded neither from below nor from above. The players can use mixed strategies.

Since the payoff function is unbounded from above and from below, the expected payoffs are uncertain for some pairs of mixed strategies. This article provides sufficient conditions for the existence of a solution or, in other words, for the existence of a pair of equilibrium strategies; see Theorem 3.9 below. Each strategy in an equilibrium pair satisfies the following property: if the corresponding player chooses this strategy, then the expected finite or infinite payoff is defined for every strategy played by the opponent. Strategies of each player with this property are called sensible. Of course, if the payoff functions are bounded from below or from above, then every strategy is sensible. However, if the decision sets for both players are noncompact, then natural mild continuity conditions (see Assumption 2.6 (a2, b2) below) imply that the payoff function is unbounded from below and from above. This article also provides sufficient conditions for the existence of values and solutions for each player.

The currently available general result on the existence of solutions for infinite games, Mertens et al. (2015, Theorem I.2.4), assumes that both decision sets are compact and the payoff function is bounded from below or above. Its proof in Mertens et al. (2015) uses Sion's minimax theorem for quasiconvex/quasiconcave functions applied to the sets of mixed strategies of the original game. This method works only if the payoff function is bounded from below or from above. If the payoff function is bounded neither from below nor from above, the expected payoffs are not defined for some pairs of mixed strategies. Sion's minimax theorem assumes that decision sets are subsets of linear topological spaces, and one of them is compact. There are minimax theorems for more general spaces, than linear topological spaces, see Khanh and Quan (2010) and Park (2013), but they also assume that one of decision sets is compact; see, for example, Khanh and Quan (2010, Corollary 4.1).

Our proofs are based on the lopsided minimax equality established in Feinberg et al. (2022, Theorem 18) for games with two possibly noncompact decision sets; see Theorem 2.7 (a) below. The classic sufficient conditions for the validity of such minimax equalities require that one of the decision sets is compact; see Mertens et al. (2015, Propositions I.1.9 and I.2.2) and Alpern and Gal (1988). Theorem 2.7 leads to Assumptions 2.8 and 3.5 introduced in Sections 2 and 3. These assumptions are equivalent to the assumption that, if one of the players chooses an unsensible strategy, another player can respond with a strategy causing the infinite loss

to the player who chose the unsensible strategy. The lopsided minimax equality in Feinberg et al. (2022, Theorem 18) (see Theorem 2.7 (a) below) is used in Feinberg et al. (2022, Theorem 20) to prove the existence of a solution if payoffs are bounded from below.

Section 2 of this article provides main definitions and preliminary results, some of which are taken from our previous paper Feinberg et al. (2022). Section 3 provides main results. In addition to Assumptions 2.8 and 3.5 mentioned above, the existence of a solution is established in Theorem 3.9 under the conditions that the payoff function is lower/upper semi-continuous in the corresponding variables, and the payoff function is inf/sup-compact in the decision variable corresponding to decisions of the player for at least one decision chosen by another player. The same conditions are assumed for the payoff function in Aubin and Ekeland (1984, Theorem 6.2.8) stating the existence of solutions within the class of pure strategies for convex/concave payoff functions. The results are illustrated in Section 4 in which the following game is considered. Two players choose nonnegative numbers, and the payoff is a polynomial function of the difference between these numbers. We completely classify this game: in some cases there are no solutions, in some cases solutions exist, an in some cases solutions exist, but pure solutions do not exist. Some of the proofs are provided in Section 5.

Our initial motivation for studying games with unbounded payoffs in Feinberg et al. (2022) and in this article was originated by the progress in the theory of Markov decision processes with possibly noncompact decision sets and unbounded costs that led to the extension of Berge's maximum theorem, see Berge (1963, p. 116), to possibly noncompact decision sets; see Feinberg and Kasyanov (2015), Feinberg, Kasyanov, and Voorneveld (2014), Feinberg, Kasyanov, and Zgurovsky (2014), and Feinberg et al. (2013, 2021). These results were applied in Feinberg et al. (2022) to games with perfect information and, as described above, some results for games with simultaneous moves are also obtained in Feinberg et al. (2022). In particular, the existence of solutions is established in Feinberg et al. (2022, Theorem 20) for payoffs bounded below, and this and other assumptions imply compactness of one of the decision sets. This article studies more general models, when both decision sets may be noncompact. These models have potential applications to stochastic games; see Jaśkiewicz and Nowak (2011, 2018), Mertens et al. (2015, Chapter VII), and references therein for the literature on stochastic games.

2 | DEFINITIONS AND PRELIMINARY RESULTS

Let $\mathbb{S}$ be a metric space, and $\mathcal{B}(\mathbb{S})$ be the Borel σ-field on $\mathbb{S}$, that is, the σ-field is generated by all open sets of the metric space $\mathbb{S}$. For a nonempty Borel subset $S \subset \mathbb{S}$, let $\mathcal{B}(S)$ denote the σ-field whose elements are intersections of S with elements of $\mathcal{B}(\mathbb{S})$. Observe that S is a metric space with the same metric as on $\mathbb{S}$. Therefore, $\mathcal{B}(S)$ is its Borel σ-field. Let $\mathbb{P}(\mathbb{S})$ be the set of probability measures on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$. We denote by $\mathbb{P}^{fs}(S)$ the set of all probability measures whose supports are finite subsets of the set S. A sequence of probability measures $\{\mu^{(n)}\}_{n=1,2,\ldots}$ from $\mathbb{P}(\mathbb{S})$ *converges weakly* to $\mu \in \mathbb{P}(\mathbb{S})$ if for each bounded continuous function f on $\mathbb{S}$

$$\int_{\mathbb{S}} f(s)\mu^{(n)}(ds) \to \int_{\mathbb{S}} f(s)\mu(ds) \qquad \text{as} \quad n \to \infty.$$

We endow $\mathbb{P}(\mathbb{S})$ with the topology of the weak convergence of probability measures on $\mathbb{S}$. If $\mathbb{S}$ is a separable metric space, then $\mathbb{P}(\mathbb{S})$ is separable metric space too and the set $\mathbb{P}^{fs}(\mathbb{S})$ is dense in $\mathbb{P}(\mathbb{S})$; Parthasarathy (1967, Chapter II, Theorems 6.2 and 6.3). Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, where $\mathbb{R}$ is the set of real numbers.

An integral $\int_{\mathbb{S}} f(s)\mu(ds)$ of a measurable $\overline{\mathbb{R}}$-valued function f on $\mathbb{S}$ over the measure $\mu \in \mathbb{P}(\mathbb{S})$ is well-defined if either $-\infty < \int_{\mathbb{S}} f^-(s)\mu(ds)$ or $\int_{\mathbb{S}} f^+(s)\mu(ds) < +\infty$, where $f^-(s) = \min\{f(s), 0\}$, $f^+(s) = \max\{f(s), 0\}$, $s \in \mathbb{S}$. If the integral is well-defined, then $\int_{\mathbb{S}} f(s)\mu(ds) := \int_{\mathbb{S}} f^+(s)\mu(ds) + \int_{\mathbb{S}} f^-(s)\mu(ds)$.

Definition 2.1. A *two-person zero-sum game* is a triplet $\{\mathbb{A}, \mathbb{B}, c\}$, where

(i) $\mathbb{A}$ is the *space of decisions for Player I*, which is a nonempty Borel subset of a Polish space;

(ii) $\mathbb{B}$ is the *space of decisions for Player II*, which is a nonempty Borel subset of a Polish space;

(iii) the *payoff* from Player I to Player II, $-\infty < c(a, b) < +\infty$, for choosing decisions $a \in \mathbb{A}$ and $b \in \mathbb{B}$, is a *measurable* function on $\mathbb{A} \times \mathbb{B}$;

(iv) for each $b \in \mathbb{B}$ the function $a \mapsto c(a, b)$ is bounded from below on $\mathbb{A}$;

(v) for each $a \in \mathbb{A}$ the function $b \mapsto c(a, b)$ is bounded from above on $\mathbb{B}$.

The game is played as follows:

- a decision-makers (Players I and II) choose simultaneously respective decisions $a \in \mathbb{A}$ and $b \in \mathbb{B}$;
- the result (a, b) is announced to both of them;
- Player I pays Player II the amount $c(a, b)$.

Everywhere in this article, except in some pathological situations described in Section 4, we assume that a game $\{\mathbb{A}, \mathbb{B}, c\}$ satisfies conditions (i–v) from Definition 2.1 and consider only such games.

Strategies (sometimes called "mixed strategies") for Players I and II are probability measures $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$ and $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$ respectively. Moreover, $\pi^{\mathbb{A}}$ ($\pi^{\mathbb{B}}$) is called *pure* if the probability measure $\pi^{\mathbb{A}}(\cdot)$ ($\pi^{\mathbb{B}}(\cdot)$) is concentrated at one

point. Note that $\mathbb{P}(\mathbb{A})$ is the *set of strategies* for Player I, and $\mathbb{P}(\mathbb{B})$ is the *set of strategies* for Player II.

With a slight abuse of notation, we identify a pure strategy with the decision it chooses. In particular, $\mathbb{A}$ and $\mathbb{B}$ are the sets of pure strategies for Players I and II respectively. We usually write a instead of $\delta_{\{a\}}$ and b instead of $\delta_{\{b\}}$, where $\delta_{\{a\}}$ and $\delta_{\{b\}}$ are probability measures on $(\mathbb{A}, \mathcal{B}(\mathbb{A}))$ and $(\mathbb{B}, \mathcal{B}(\mathbb{B}))$ concentrated at the points $a \in \mathbb{A}$ and $b \in \mathbb{B}$ respectively.

Let

$$\hat{c}^{\boxplus}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) := \int_{\mathbb{A}} \int_{\mathbb{B}} c^+(a,b) \pi^{\mathbb{B}}(db) \pi^{\mathbb{A}}(da), \quad \hat{c}^{\boxminus}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$$
$$:= \int_{\mathbb{A}} \int_{\mathbb{B}} c^-(a,b) \pi^{\mathbb{B}}(db) \pi^{\mathbb{A}}(da),$$

for each $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}(\mathbb{A}) \times \mathbb{P}(\mathbb{B})$. Then the *expected payoff* from Player I to Player II is

$$\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) := \hat{c}^{\boxplus}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) + \hat{c}^{\boxminus}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}), \qquad (2.1)$$

and it is well-defined, if either $\hat{c}^{\boxplus}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) < +\infty$ or $-\infty < \hat{c}^{\boxminus}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$, where $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}(\mathbb{A}) \times \mathbb{P}(\mathbb{B})$. We notice that it is possible that $\hat{c}^{\boxplus} \neq \hat{c}^+$ and $\hat{c}^{\boxminus} \neq \hat{c}^-$. Of course, when the function c is unbounded from below and from above, the quantity $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$ is *undefined* for some $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}(\mathbb{A}) \times \mathbb{P}(\mathbb{B})$. Assumptions (iv) and (v) for the game $\{\mathbb{A}, \mathbb{B}, c\}$ imply that $-\infty < \hat{c}^{\boxminus}(\pi^{\mathbb{A}}, b)$ for each $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$ and $b \in \mathbb{B}$, and $\hat{c}^{\boxplus}(a, \pi^{\mathbb{B}}) < +\infty$ for each $a \in \mathbb{A}$ and $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$ respectively.

The set of strategies for each player is partitioned into the sets of *sensible strategies* $\mathbb{P}^S(\mathbb{A})$ and $\mathbb{P}^S(\mathbb{B})$ (strategies, for which the expected payoff is well-defined for all strategies played by another player) and *unsensible strategies* $\mathbb{P}^U(\mathbb{A})$ and $\mathbb{P}^U(\mathbb{B})$:

According to Feinberg et al. (2022, Theorem 17 and Lemma 6),

$$\hat{c}^{\flat}(\pi^{\mathbb{B}}) = \inf_{\pi_*^{\mathbb{A}} \in \mathbb{P}^S(\mathbb{A})} \hat{c}(\pi_*^{\mathbb{A}}, \pi^{\mathbb{B}}) \quad \text{and} \quad \hat{c}^{\sharp}(\pi^{\mathbb{A}})$$
$$= \sup_{\pi_*^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})} \hat{c}(\pi^{\mathbb{A}}, \pi_*^{\mathbb{B}}), \qquad (2.4)$$

the inequality

$$\hat{c}^{\flat}(\pi^{\mathbb{B}}) \leq \hat{c}^{\sharp}(\pi^{\mathbb{A}}), \qquad (2.5)$$

holds for each $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$ and $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$ such that $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$ is well-defined, the functions $\hat{c}^{\sharp}$ and $\hat{c}^{\flat}$ are convex on $\mathbb{P}(\mathbb{A})$ and concave on $\mathbb{P}(\mathbb{B})$ respectively, and the following level sets

$$\mathbb{P}_{\alpha}^{\sharp}(\mathbb{A}) := \{\pi_*^{\mathbb{A}} \in \mathbb{P}(\mathbb{A}) \; : \; \hat{c}^{\sharp}(\pi_*^{\mathbb{A}}) \leq \alpha\},$$
$$\mathbb{P}_{\alpha}^{\flat}(\mathbb{B}) := \{\pi_*^{\mathbb{B}} \in \mathbb{P}(\mathbb{B}) \; : \; \hat{c}^{\flat}(\pi_*^{\mathbb{B}}) \geq \alpha\},$$

are convex for all $\alpha \in \overline{\mathbb{R}}$.

In particular, $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$ is well-defined for $\pi^{\mathbb{A}} \in \mathbb{P}^S(\mathbb{A})$ and $\pi^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})$. Therefore, inequality (2.5) holds for $\pi^{\mathbb{A}} \in \mathbb{P}^S(\mathbb{A})$ and $\pi^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})$. It is not clear whether inequality (2.5) holds for $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$ and $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$ when $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$ is undefined.

The following definition introduces the lower and upper values in a slightly more delicate way than it is usually done when $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$ are always defined.

Definition 2.2 (Lower and upper values of the game). The quantities

$$v^{\flat} := \sup_{\pi^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})} \hat{c}^{\flat}(\pi^{\mathbb{B}}) \quad \text{and} \quad v^{\sharp} := \inf_{\pi^{\mathbb{A}} \in \mathbb{P}^S(\mathbb{A})} \hat{c}^{\sharp}(\pi^{\mathbb{A}}), \quad (2.6)$$

are the *lower and upper values of the game* $\{\mathbb{A}, \mathbb{B}, c\}$ respectively.

Since inequality (2.5) is valid for every pair of strategies $\pi^{\mathbb{A}} \in \mathbb{P}^S(\mathbb{A})$ and $\pi^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})$, the inequality $v^{\flat} \leq v^{\sharp}$ holds.

$$\mathbb{P}^S(\mathbb{A}) := \{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A}) \; : \; \hat{c}(\pi^{\mathbb{A}}, \pi_*^{\mathbb{B}}) \text{ is well-defined for each } \pi_*^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})\},$$
$$\mathbb{P}^U(\mathbb{A}) := \{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A}) \; : \; \hat{c}(\pi^{\mathbb{A}}, \pi_*^{\mathbb{B}}) \text{ is undefined for some } \pi_*^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})\},$$
$$\mathbb{P}^S(\mathbb{B}) := \{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B}) \; : \; \hat{c}(\pi_*^{\mathbb{A}}, \pi^{\mathbb{B}}) \text{ is well-defined for each } \pi_*^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})\},$$
$$\mathbb{P}^U(\mathbb{B}) := \{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B}) \; : \; \hat{c}(\pi_*^{\mathbb{A}}, \pi^{\mathbb{B}}) \text{ is undefined for some } \pi_*^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})\}.$$

Assumptions (iv) and (v) from Definition 2.1 imply that $\mathbb{P}^{fs}(\mathbb{A}) \subset \mathbb{P}^S(\mathbb{A})$ and $\mathbb{P}^{fs}(\mathbb{B}) \subset \mathbb{P}^S(\mathbb{B})$. Therefore, $\mathbb{P}^S(\mathbb{A})$ is dense in $\mathbb{P}(\mathbb{A})$ and $\mathbb{P}^S(\mathbb{B})$ is dense in $\mathbb{P}(\mathbb{B})$.

Let us introduce the *worst gain* of Player II for choosing a strategy $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$ and *worst loss* of Player I for choosing a strategy $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$ respectively:

$$\hat{c}^{\flat}(\pi^{\mathbb{B}}) := \inf_{a \in \mathbb{A}} \hat{c}(a, \pi^{\mathbb{B}}) \quad \text{and} \quad \hat{c}^{\sharp}(\pi^{\mathbb{A}}) := \sup_{b \in \mathbb{B}} \hat{c}(\pi^{\mathbb{A}}, b). \qquad (2.2)$$

These definitions and assumptions (iv,v) from Definition 2.1 of the game $\{\mathbb{A}, \mathbb{B}, c\}$ imply

$$-\infty < \hat{c}^{\flat}(b) \leq c(a,b) \leq \hat{c}^{\sharp}(a) < +\infty, \qquad a \in \mathbb{A}, \; b \in \mathbb{B}, \qquad (2.3)$$

In addition, $-\infty < \sup_{b \in \mathbb{B}} \hat{c}^{\flat}(b) \leq \sup_{\pi^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})} \hat{c}^{\flat}(\pi^{\mathbb{B}}) = v^{\flat}$, where the second inequality holds because every pure strategy is sensible in view of assumptions (iv) and (v) from Definition 2.1, and the first inequality follows from (2.3). Similarly, $v^{\sharp} < +\infty$. Therefore,

$$-\infty < v^{\flat} \leq v^{\sharp} < +\infty. \qquad (2.7)$$

Equalities (2.4) and (2.6) imply

$$v^{\flat} = \sup_{\pi^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})} \inf_{\pi^{\mathbb{A}} \in \mathbb{P}^S(\mathbb{A})} \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \quad \text{and}$$
$$v^{\sharp} = \inf_{\pi^{\mathbb{A}} \in \mathbb{P}^S(\mathbb{A})} \sup_{\pi^{\mathbb{B}} \in \mathbb{P}^S(\mathbb{B})} \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}). \qquad (2.8)$$

If $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$ is well-defined for each $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}(\mathbb{A}) \times \mathbb{P}(\mathbb{B})$, as this takes place when c is bounded from below or above on

$\mathbb{A} \times \mathbb{B}$, then the upper and lower values of the game defined in (2.6) coincide with their classic definition, that is,

$$v^\flat = \sup_{\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})} \inf_{\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})} \hat{c}(\pi^\mathbb{A}, \pi^\mathbb{B}) \quad \text{and}$$

$$v^\sharp = \inf_{\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})} \sup_{\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})} \hat{c}(\pi^\mathbb{A}, \pi^\mathbb{B}). \tag{2.9}$$

Definition 2.3 (Solution for a player). A strategy $\pi^\mathbb{A} \in \mathbb{P}^S(\mathbb{A})$ ($\pi^\mathbb{B} \in \mathbb{P}^S(\mathbb{B})$) is called *a solution for Player I (II)* if

$$\hat{c}^\sharp(\pi^\mathbb{A}) = v^\sharp \quad (\hat{c}^\flat(\pi^\mathbb{B}) = v^\flat). \tag{2.10}$$

A solution $\pi^\mathbb{A}$ ($\pi^\mathbb{B}$) for Player I (II) is called *pure* if the respective strategy $\pi^\mathbb{A}$ ($\pi^\mathbb{B}$) is pure.

Definition 2.4 (Value of a game). If the equality

$$v^\flat = v^\sharp, \tag{2.11}$$

holds, then we say that this common quantity is the *value of the game* $\{\mathbb{A}, \mathbb{B}, c\}$. We denote the value by v.

In view of this definition, if the value v exists, then it is unique and, in view of (2.7), v is a real number. If $\hat{c}(\pi^\mathbb{A}, \pi^\mathbb{B})$ is well-defined for each $(\pi^\mathbb{A}, \pi^\mathbb{B}) \in \mathbb{P}(\mathbb{A}) \times \mathbb{P}(\mathbb{B})$, as this takes place if and only if c is bounded from below or above on $\mathbb{A} \times \mathbb{B}$, then the value defined in (2.11) coincides with its classic definition, that is,

$$v = v^\flat = \sup_{\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})} \hat{c}^\flat(\pi^\mathbb{B}) = \sup_{\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})} \inf_{\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})} \hat{c}(\pi^\mathbb{A}, \pi^\mathbb{B}) = v^\sharp$$

$$= \inf_{\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})} \hat{c}^\sharp(\pi^\mathbb{A}) = \inf_{\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})} \sup_{\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})} \hat{c}(\pi^\mathbb{A}, \pi^\mathbb{B}).$$

Lemma 2.5. *If the value v of a game $\{\mathbb{A}, \mathbb{B}, c\}$ exists, then*

$$v = \sup_{\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})} \hat{c}^\flat(\pi^\mathbb{B}) = \inf_{\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})} \hat{c}^\sharp(\pi^\mathbb{A}). \tag{2.12}$$

Proof. Note that

$$v^\flat = \sup_{\pi^\mathbb{B} \in \mathbb{P}^S(\mathbb{B})} \hat{c}^\flat(\pi^\mathbb{B}) \leq \sup_{\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})} \hat{c}^\flat(\pi^\mathbb{B}) \leq \inf_{\pi^\mathbb{A} \in \mathbb{P}^S(\mathbb{A})} \hat{c}^\sharp(\pi^\mathbb{A}) = v^\sharp, \tag{2.13}$$

where the equalities follow from the definitions of lower and upper values (2.6), the first inequality follows from $\mathbb{P}^S(\mathbb{B}) \subset \mathbb{P}(\mathbb{B})$, and the second inequality follows from inequality (2.5) because $\hat{c}(\pi^\mathbb{A}, \pi^\mathbb{B})$ is well-defined for each pair $(\pi^\mathbb{A}, \pi^\mathbb{B}) \in \mathbb{P}^S(\mathbb{A}) \times \mathbb{P}(\mathbb{B})$. Similarly,

$$v^\flat = \sup_{\pi^\mathbb{B} \in \mathbb{P}^S(\mathbb{B})} \hat{c}^\flat(\pi^\mathbb{B}) \leq \inf_{\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})} \hat{c}^\sharp(\pi^\mathbb{A}) \leq \inf_{\pi^\mathbb{A} \in \mathbb{P}^S(\mathbb{A})} \hat{c}^\sharp(\pi^\mathbb{A}) = v^\sharp. \tag{2.14}$$

Therefore, (2.12) follows from (2.13) and (2.14) because $v^\flat = v^\sharp = v$. ∎

We recall that for a metric space $\mathbb{S}$ a function $f : \mathbb{S} \to \overline{\mathbb{R}}$ is called *lower semi-continuous at $s \in \mathbb{S}$*, if for each sequence $\{s^{(n)}\}_{n=1,2,\ldots}$, that converges to s in $\mathbb{S}$, the inequality $\liminf_{n\to\infty} f(s^{(n)}) \geq f(s)$ holds. A function $f : \mathbb{S} \to \overline{\mathbb{R}}$

is called *upper semi-continuous at $s \in \mathbb{S}$*, if $-f$ is lower semi-continuous at $s \in \mathbb{S}$. A function $f : \mathbb{S} \to \overline{\mathbb{R}}$ is called *lower/upper semi-continuous*, if f is lower/upper semi-continuous at each $s \in \mathbb{S}$. A function $f : \mathbb{S} \to \overline{\mathbb{R}}$ is called *inf-compact on $\mathbb{S}$*, if all the level sets $\{\{s \in \mathbb{S} : f(s) \leq \lambda\}\}_{\lambda \in \mathbb{R}}$ are compact in $\mathbb{S}$. A function $f : \mathbb{S} \to \overline{\mathbb{R}}$ is called *sup-compact on $\mathbb{S}$* if $-f$ is inf-compact on $\mathbb{S}$.

The following assumptions contains all the semi-continuity assumptions on the payoff function c used in this article.

Assumption 2.6. The payoff function $c : \mathbb{A} \times \mathbb{B} \to \mathbb{R}$ satisfies the following conditions:

(a1) for each $b \in \mathbb{B}$ the function $a \mapsto c(a, b)$ is lower semi-continuous;

(a2) there exists $b_0 \in \mathbb{B}$ such that the function $a \mapsto c(a, b_0)$ is inf-compact on $\mathbb{A}$;

(b1) for each $a \in \mathbb{A}$ the function $b \mapsto c(a, b)$ is upper semi-continuous;

(b2) there exists $a_0 \in \mathbb{A}$ such that the function $b \mapsto c(a_0, b)$ is sup-compact on $\mathbb{B}$.

The following theorem, whose part (a) is taken from Feinberg et al. (2022, Subsection 5.4), provides sufficient conditions for the existence of a value of the game $\{\mathbb{A}, \mathbb{B}, c\}$ in the sense of Definition 2.4.

Theorem 2.7.

(a) (Existence of a lopsided value; Feinberg et al. (2022, Theorem 18 and Corollary 4)). If a two-person zero-sum game $\{\mathbb{A}, \mathbb{B}, c\}$ satisfies Assumption 2.6 (a1,a2), then

$$\min_{\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})} \hat{c}^\sharp(\pi^\mathbb{A}) = \sup_{\pi^\mathbb{B} \in \mathbb{P}^{fs}(\mathbb{B})} \hat{c}^\flat(\pi^\mathbb{B}) = v^\flat, \tag{2.15}$$

and the set $\mathbb{P}^\sharp_{v^\flat}(\mathbb{A})$ is a nonempty convex compact subset of $\mathbb{P}(\mathbb{A})$.

(b) (Existence of the value). Under the assumptions of statement (a), the value $v = v^\flat = v^\sharp$ of the game exists if and only if

$$v^\sharp = \inf_{\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})} \hat{c}^\sharp(\pi^\mathbb{A}). \tag{2.16}$$

Proof of Theorem 2.7(b). Let the assumptions of Theorem 2.7 (a) hold. In view of (2.15), the infimum in (2.16) can be replaced with the minimum. If $v^\flat = v^\sharp$, then (2.15) implies (2.16). If (2.16) holds, then (2.15) implies $v^\flat = v^\sharp$. ∎

Theorem 2.7 (b) is useful for proving the existence of the value. Observe that (2.16) holds under the following condition:

Assumption 2.8. $\hat{c}^\sharp(\pi^\mathbb{A}) > v^\sharp$ for all $\pi^\mathbb{A} \in \mathbb{P}^U(\mathbb{A})$.

This is true because $\mathbb{P}(\mathbb{A}) = \mathbb{P}^S(\mathbb{A}) \cup \mathbb{P}^U(\mathbb{A})$, and, if Assumption 2.8 is correct, then (2.16) becomes the definition of $v^\sharp$ given in (2.6). Therefore, Assumption 2.8 and Assumption 2.6 (a1,a2) imply the existence of the value of

the game. Moreover, if the value v exists and Assumption 2.8 holds, then

$$\mathbb{P}_v^{\sharp}(\mathbb{A}) \subset \mathbb{P}^S(\mathbb{A}). \tag{2.17}$$

The similar observations are applicable to the lower value $v^{\flat}$, when condition (2.18) is replaced symmetrically with condition (3.9) for Player II presented below.

Note that Assumption 2.8 holds if

$$\hat{c}^{\sharp}(\pi^{\mathbb{A}}) = +\infty \quad \text{for all } \pi^{\mathbb{A}} \in \mathbb{P}^U(\mathbb{A}). \tag{2.18}$$

However, as Example 3 in Feinberg et al. (2022) demonstrates, it is possible that $\hat{c}^{\sharp}(\pi^{\mathbb{A}}) < +\infty$ for some $\pi^{\mathbb{A}} \in \mathbb{P}^U(\mathbb{A})$ even under stronger conditions than Assumption 2.6 (a1,a2). We also note that statement (2.18) holds if and only if $\mathbb{P}_{\alpha}^{\sharp}(\mathbb{A}) \subset \mathbb{P}^S(\mathbb{A})$ for each $\alpha \in \mathbb{R}$.

> *Remark* 2.9. Assumption 2.6 (a1,a2) are natural for the existence of a solution for Player I. For example, they are assumed in Aubin and Ekeland (1984, Theorem 6.2.7), where the existence of a solution for Player I is stated for payoff functions $c(a, b)$ being convex in a and concave in b. In particular, if the decision set of Player II is a singleton, that is, $\mathbb{B} = \{b_0\}$, then the game becomes an optimization problem. For this optimization problem, inf-compactness Assumption 2.6 (a2) is a natural sufficient condition for the existence of a minimum, and this minimum corresponds to the solution for Player I and for the game. If the function c is bounded above, then Assumption 2.6 (a2) implies that the set $\mathbb{A}$ is compact because $\mathbb{A} = \{a \in \mathbb{A} : c(a, b_0) \le \lambda\}$ for some $\lambda \in \mathbb{R}$. If the set $\mathbb{A}$ is compact, then Assumption 2.6 (a1) implies Assumption 2.6 (a2).

Let us consider some corollaries from Theorem 2.7. We recall that, as discussed in Section 2, the action sets $\mathbb{A}$ and $\mathbb{B}$ can be identified with the sets of pure strategies.

> **Corollary 2.10.** *(a) For is a two-person game $\{\mathbb{A}, \mathbb{B}, c\}$,*
>
> $$\hat{c}^{\sharp}(\pi^{\mathbb{A}}) = \sup_{\pi^{\mathbb{B}} \in \Delta'(\mathbb{B})} \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}), \qquad \pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A}), \tag{2.19}$$
>
> *for all $\Delta'(\mathbb{B}) \subset \mathbb{P}(\mathbb{B})$ such that $\mathbb{B} \subset \Delta'(\mathbb{B}) \subset \mathbb{P}^S(\mathbb{B})$.*
>
> *(b) (Feinberg et al., 2022, Corollary 3). If Assumption 2.6 (a1,a2) hold, then*
>
> $$\sup_{\pi^{\mathbb{B}} \in \Delta(\mathbb{B})} \hat{c}^{\flat}(\pi^{\mathbb{B}}) = \min_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}^{\sharp}(\pi^{\mathbb{A}}),$$
>
> *for all $\Delta(\mathbb{B}) \subset \mathbb{P}(\mathbb{B})$ such that $\mathbb{P}^{fs}(\mathbb{B}) \subset \Delta(\mathbb{B}) \subset \mathbb{P}^S(\mathbb{B})$.*
>
> *Proof.* Statement (a) follows from (2.2) and (2.4). ∎

The following corollary states the classic minimax equality, which is well-known for a compact set $\mathbb{A}$; see Mertens et al. (2015, Proposition I.1.9).

> **Corollary 2.11** (Feinberg et al. (2022, Corollary 5)). *If for each $b \in \mathbb{B}$ the function $a \mapsto c(a, b)$ is inf-compact, then*
>
> $$\min_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \sup_{\pi^{\mathbb{B}} \in \mathbb{P}^{fs}(\mathbb{B})} \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) = \sup_{\pi^{\mathbb{B}} \in \mathbb{P}^{fs}(\mathbb{B})} \min_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}). \tag{2.20}$$

Corollary 2.10 implies additional versions of equality (2.20) with the set $\mathbb{P}^{fs}(\mathbb{B})$ in the left and right hand sides of (2.20) replaced with arbitrary sets $\Delta'(\mathbb{B})$ and $\Delta(\mathbb{B})$ specified in statements (a) and (b) of Corollary 2.10 respectively. In addition, according to Feinberg et al. (2022, Theorem 17), equality (2.19) holds when $\mathbb{B} \subset \Delta'(\mathbb{B}) \subset \mathbb{P}_{\pi^{\mathbb{A}}}^S(\mathbb{B})$, where $\mathbb{P}_{\pi^{\mathbb{A}}}^S(\mathbb{B}) = \{\pi_*^{\mathbb{B}} \in \mathbb{P}(\mathbb{B}) : \hat{c}(\pi^{\mathbb{A}}, \pi_*^{\mathbb{B}})$ is well-defined$\}$, $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$. Therefore, in the left hand side of (2.20) the set $\mathbb{P}^{fs}(\mathbb{B})$ can be replaced with a set Δ' such that $\mathbb{B} \subset \Delta'(\mathbb{B}) \subset \mathbb{P}_{\pi^{\mathbb{A}}}^S(\mathbb{B})$.

Next we define solutions for a game with a payoff function unbounded from above and from below.

> **Definition 2.12** (Solution for a game). A pair of strategies $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}^S(\mathbb{A}) \times \mathbb{P}^S(\mathbb{B})$ for Players I and II is called a *solution (saddle point, equilibrium) of the game* $\{\mathbb{A}, \mathbb{B}, c\}$ if
>
> $$\hat{c}(\pi^{\mathbb{A}}, \pi_*^{\mathbb{B}}) \le \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \le \hat{c}(\pi_*^{\mathbb{A}}, \pi^{\mathbb{B}}), \tag{2.21}$$
>
> for all $\pi_*^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$ and $\pi_*^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$. A solution $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$ for the game $\{\mathbb{A}, \mathbb{B}, c\}$ is called *pure* if the strategies $\pi^{\mathbb{A}}$ and $\pi^{\mathbb{B}}$ are pure.

If a solution $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}^S(\mathbb{A}) \times \mathbb{P}^S(\mathbb{B})$ for the game $\{\mathbb{A}, \mathbb{B}, c\}$ exists, the real number $v = \hat{c}^{\flat}(\pi^{\mathbb{B}}) = v^{\flat} = v^{\sharp} = \hat{c}^{\sharp}(\pi^{\mathbb{A}}) = \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$ is the value of this game. Indeed, since $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}^S(\mathbb{A}) \times \mathbb{P}^S(\mathbb{B})$, inequalities (2.21) imply that $\hat{c}^{\flat}(\pi^{\mathbb{B}}) = \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) = \hat{c}^{\sharp}(\pi^{\mathbb{A}})$. Therefore,

$$v^{\flat} = \hat{c}^{\flat}(\pi^{\mathbb{B}}) = \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) = \hat{c}^{\sharp}(\pi^{\mathbb{A}}) = v^{\sharp}, \tag{2.22}$$

where the first and the last equalities hold since $\hat{c}^{\flat}(\pi_*^{\mathbb{B}}) \le \hat{c}^{\sharp}(\pi^{\mathbb{A}})$ and $\hat{c}^{\flat}(\pi^{\mathbb{B}}) \le \hat{c}^{\sharp}(\pi_*^{\mathbb{A}})$ for all $(\pi_*^{\mathbb{A}}, \pi_*^{\mathbb{B}}) \in \mathbb{P}^S(\mathbb{A}) \times \mathbb{P}^S(\mathbb{B})$.

Moreover, $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}^S(\mathbb{A}) \times \mathbb{P}^S(\mathbb{B})$ is the solution for the game $\{\mathbb{A}, \mathbb{B}, c\}$ if and only if there exists the value v, the strategy $\pi^{\mathbb{A}}$ is a solution for Player I, and the strategy $\pi^{\mathbb{B}}$ is a solution for Player II. Indeed, the necessary condition is proved in the previous paragraph, and the sufficient condition follows from $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \le \hat{c}^{\sharp}(\pi^{\mathbb{A}}) = v = \hat{c}^{\flat}(\pi^{\mathbb{B}}) \le \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$, where the equality in the middle hold because $\pi^{\mathbb{A}}$ and $\pi^{\mathbb{B}}$ are solutions for Players I and II respectively, and v is the value.

If a pure solution $(a, b) \in \mathbb{A} \times \mathbb{B}$ for the game $\{\mathbb{A}, \mathbb{B}, c\}$ exists, then the number

$$v = \hat{c}^{\flat}(b) = v^{\flat} = v^{\sharp} = \hat{c}^{\sharp}(a) = c(a, b), \tag{2.23}$$

is the value of this game. Moreover,

$$v = \sup_{b^* \in \mathbb{B}} \inf_{a^* \in \mathbb{A}} c(a^*, b^*) = \inf_{a^* \in \mathbb{A}} \sup_{b^* \in \mathbb{B}} c(a^*, b^*),$$

because, by the definitions of $\hat{c}^{\flat}$ and $\hat{c}^{\sharp}$,

$$\sup_{b^* \in \mathbb{B}} \inf_{a^* \in \mathbb{A}} c(a^*, b^*) = \sup_{b^* \in \mathbb{B}} \hat{c}^{\flat}(b^*),$$

$$\inf_{a^* \in \mathbb{A}} \sup_{b^* \in \mathbb{B}} c(a^*, b^*) = \inf_{a^* \in \mathbb{A}} \hat{c}^{\sharp}(a^*),$$

and, in view of (2.23),

$$\inf_{a^* \in \mathbb{A}} \hat{c}^\sharp(a^*) \le \hat{c}^\sharp(a) = v = \hat{c}^\flat(b) \le \sup_{b^* \in \mathbb{B}} \hat{c}^\flat(b^*) \le \inf_{a^* \in \mathbb{A}} \hat{c}^\sharp(a^*),$$

where the last inequality follows from (2.5) with $\pi^\mathbb{A} = \delta_{\{a\}}$ and $\pi^\mathbb{B} = \delta_{\{b\}}$. Therefore, the game $\{\mathbb{A}, \mathbb{B}, c\}$ has a solution in pure strategies (that is, the players can play only pure strategies, and this game has a solution) if and only if there is a pure solution for the game $\{\mathbb{A}, \mathbb{B}, c\}$.

Remark 2.13. Let $\{\mathbb{A}, \mathbb{B}, c\}$ be a two-person zero-sum game introduced in Definition 2.1. Then the triplet $\{\mathbb{B}, \mathbb{A}, -c^{\mathbb{A} \leftrightarrow \mathbb{B}}\}$, where $c^{\mathbb{A} \leftrightarrow \mathbb{B}}(b, a) := c(a, b)$ for each $a \in \mathbb{A}$ and $b \in \mathbb{B}$, is also a game satisfying conditions (i–v) from Definition 2.1. If this construction is repeated, it leads to the original game. The game $\{\mathbb{A}, \mathbb{B}, c\}$ has a value (solution for Player I, Player II, solution) if and only if the game $\{\mathbb{B}, \mathbb{A}, -c^{\mathbb{A} \leftrightarrow \mathbb{B}}\}$ has a value (solution for Player II, Player I, solution).

3 | MAIN RESULTS AND DISCUSSION

Since this article deals with symmetrically defined games, each assumption or statement for Player I can be reformulated as the corresponding assumption or statement for Player II. In this article, we provide assumptions and statements that involve both Players or only Player I. Symmetric assumptions and statements for Player II are provided only if they are used in explanations or in other statements.

To provide sufficient conditions for the validity of Assumption 2.8, let us define

$$\hat{c}^{\boxplus,\sharp}(\pi^\mathbb{A}) := (\hat{c}^\boxplus)^\sharp(\pi^\mathbb{A}) = \sup_{b \in \mathbb{B}} \hat{c}^\boxplus(\pi^\mathbb{A}, b), \qquad \pi^\mathbb{A} \in \mathbb{P}(\mathbb{A}). \tag{3.1}$$

We also define the symmetric function for Player II,

$$\hat{c}^{\boxminus,\flat}(\pi^\mathbb{B}) := (\hat{c}^\boxminus)^\flat(\pi^\mathbb{B}) = \inf_{a \in \mathbb{A}} \hat{c}^\boxminus(a, \pi^\mathbb{B}), \qquad \pi^\mathbb{B} \in \mathbb{P}(\mathbb{B}). \tag{3.2}$$

The following theorem describes the sufficient conditions for Assumption 2.8.

Theorem 3.1 (Sufficient conditions for Assumption 2.8). *Assumption 2.8 holds if at least one of the following assumptions is satisfied:*

(L) the function c is bounded below on $\mathbb{A} \times \mathbb{B}$;

(U) the function c is bounded above on $\mathbb{A} \times \mathbb{B}$;

(A1) there exist $\gamma_\mathbb{A} \in (0, 1)$, $L_\mathbb{A} > 0$, and $b_0 \in \mathbb{B}$ such that for each $a \in \mathbb{A}$

$$-L_\mathbb{A} + \gamma_\mathbb{A} \hat{c}^\sharp(a) \le c^+(a, b_0); \tag{3.3}$$

(A2) there exist $\gamma_\mathbb{A} \in (0, 1)$, $L_\mathbb{A} > 0$, and $\pi_0^\mathbb{B} \in \mathbb{P}^S(\mathbb{B})$ such that

$$-\infty < \int_\mathbb{B} \min\{0, \hat{c}^\flat(b)\} \pi_0^\mathbb{B}(db), \text{ and}$$
$$-L_\mathbb{A} + \gamma_\mathbb{A} \hat{c}^\sharp(a) \le \hat{c}^\boxplus(a, \pi_0^\mathbb{B}), \tag{3.4}$$

for each $a \in \mathbb{A}$;

(A3) there exist $\gamma_\mathbb{A} \in (0, 1)$ and $M_\mathbb{A} > 0$ such that for each $\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})$ and $\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})$

$$\gamma_\mathbb{A} \hat{c}^\boxplus(\pi^\mathbb{A}, \pi^\mathbb{B}) \le \hat{c}^\sharp(\pi^\mathbb{A}) + M_\mathbb{A}; \tag{3.5}$$

(A4) there exists a function $\Psi_\mathbb{A} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ such that $\Psi_\mathbb{A}(s) < +\infty$, if $s < +\infty$, and for each $\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})$ and $\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})$

$$\hat{c}^\boxplus(\pi^\mathbb{A}, \pi^\mathbb{B}) \le \Psi_\mathbb{A}(\hat{c}^\sharp(\pi^\mathbb{A})); \tag{3.6}$$

(A5) if $\hat{c}^\sharp(\pi^\mathbb{A}) < +\infty$ for $\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})$, then $\hat{c}^{\boxplus,\sharp}(\pi^\mathbb{A}) < +\infty$.

Moreover, Assumption (A5) is equivalent to statement (2.18).

The proof of Theorem 3.1 is provided in Section 5.

The relations between assumptions (U), (A1)–(A5) are described below in implications (5.2). According to Theorem 3.1, each of these assumption implies Assumption 2.8. However, assumptions (U), (L), (A1)–(A5) are simpler than Assumption 2.8 and useful for applications.

Assumptions (L), (U), (A1)–(A5) are formulated in terms of the primitives of the model. For example, Assumption (A5) means that for every probability $\pi^\mathbb{A}$ on the metric space $\mathbb{A}$, the inequality

$$\sup_{b \in \mathbb{B}} \left\{ \int_\mathbb{A} c^+(a, b) \pi^\mathbb{A}(da) + \int_\mathbb{A} c^-(a, b) \pi^\mathbb{A}(da) \right\} < +\infty,$$

implies

$$\sup_{b \in \mathbb{B}} \int_\mathbb{A} c^+(a, b) \pi^\mathbb{A}(da) < +\infty.$$

Simple Example 3.2, which we provide for illustrative purposes, describes a two-person zero-sum game $\{\mathbb{A}, \mathbb{B}, c\}$ with noncompact decision sets and payoff function c unbounded from above and below. In this example, expected payoffs are not defined for some pairs of strategies for Players I and II. Currently available literature does not have results on the existence of a value and a solution for such games. However, for this example it is intuitively clear that the value is 0, and pure strategies $a = 0$ and $b = 0$ for Players I and II respectively form a solution. This game satisfies assumption (A1) and, therefore, it satisfies Assumption 2.8. It also satisfies Assumption 2.6 (a1,a2,b1,b2) and, as follows from Theorem 3.7 (B1), it satisfies Assumption 3.5, which is symmetric to Assumption 2.8. In view of Theorems 2.7 (ii) and 3.9, this game has a value and a solution. In addition, the value is 0, and pure strategies $a = 0$ and $b = 0$ are the unique solutions for Player I and II respectively.

Example 3.2. Let $\mathbb{A} = \mathbb{B} = \mathbb{R}$, $c(a, b) = a^2 - b^2$, $(a, b) \in \mathbb{R}^2$. Then the game $\{\mathbb{A}, \mathbb{B}, c\}$ satisfies assumption (A1) and, therefore, it satisfies Assumption 2.8. Indeed, if we consider arbitrary $b_0 \in \mathbb{R}$, $\gamma_\mathbb{A} \in (0, 1)$ and set $L_\mathbb{A} := b_0^2$, then

$$-L_\mathbb{A} + \gamma_\mathbb{A} \hat{c}^\sharp(a) = \gamma_\mathbb{A} a^2 - b_0^2 \le a^2 - b_0^2 = c(a, b_0) \le c^+(a, b_0),$$

for each $a \in \mathbb{R}$ because $\hat{c}^\sharp(a) = a^2$ for each $a \in \mathbb{R}$.

The following Theorem 3.3 provides sufficient conditions for the existence of a value and a solution for a one of the Players for a two-person zero-sum game with possibly non-compact decision sets and unbounded payoffs. This theorem and Corollary 3.4 also describe the properties of the solution sets under these conditions. In general, an infinite game may not have a value; see Yanovskaya (1974, p. 527) and the references to counterexamples by Ville, by Wald, and by Sion and Wolfe cited there. Therefore, some additional conditions for the existence of a value and solutions are needed. The results available in the literature require among other assumptions that either at least one of the decision sets is compact (Alpern and Gal, 1988; Mertens et al., 2015, Propositions I.1.9 and I.2.2; Prokopovych and Yannelis, 2014; Tian, 1992) or the payoff function is convex/concave-like (Ansari et al., 2000; Aubin and Ekeland, 1984, Theorem 6.2.7; Nessah and Tian, 2016; Perchet and Vigeral, 2015; Zeng et al., 2006). Theorems 3.3 and 3.9 of this article require neither compactness of one of the decision sets nor convexity/concavity-like properties of the payoff function. Baye et al. (1993), Nessah and Tian (2013), and Tian (2015) provide sufficient conditions for the existence of pure solutions for games with possibly noncompact decision sets and without convexity-like assumptions. However, under the assumptions of Theorems 3.3 and 3.9, a game may have neither value nor solution when only pure strategies are played; see Proposition 4.4(a). Therefore, the results of this article do not follow from the references mentioned in this paragraph. Moreover, in games studied in this article payoffs may not be defined for some strategies chosen by players because payoff functions can be unbounded. In most of the references mentioned in this paragraph, including in Baye et al. (1993), Nessah and Tian (2013), and Tian (2015), payoffs or preference relations are defined for all strategies players can choose in the game. Note that some of these references provide significant advances to the theory of nonzero-sum games and consider relaxed versions of semi-continuity assumptions on payoff functions such as transfer lower and upper semi-continuity. However, the results are currently available either for compact decision sets or for convex/concave-like payoff functions or games with solutions in pure strategies.

Theorem 3.3 (Existence of a value and solution for Player I). *Let a two-person zero-sum game $\{\mathbb{A}, \mathbb{B}, c\}$ satisfy Assumption 2.6 (a1,a2) and*

$$\hat{c}^{\sharp}(\pi^{\mathbb{A}}) \geq v^{\sharp} \ \text{for all} \ \pi^{\mathbb{A}} \in \mathbb{P}^{U}(\mathbb{A}). \tag{3.7}$$

Then the game $\{\mathbb{A}, \mathbb{B}, c\}$ has the value v, that is, equality (2.11) and, therefore, equalities (2.12) hold. Moreover, if Assumption 2.8 holds, then the set $\mathbb{P}^{\sharp}_{v}(\mathbb{A})$, which is a subset of $\mathbb{P}^{S}(\mathbb{A})$, is the set of solutions for Player I, and $\mathbb{P}^{\sharp}_{v}(\mathbb{A})$ is a nonempty convex compact subset of $\mathbb{P}(\mathbb{A})$.

The proof of Theorem 3.3 is provided in Section 5.

Corollary 3.4 (Compactness of the set $\mathbb{P}^{\flat}_{v}(\mathbb{B})$). *Let the assumptions of Theorem 3.3 and Assumption 2.6 (b1, b2) hold. Then, in addition to the conclusions of Theorem 3.3, the set $\mathbb{P}^{\flat}_{v}(\mathbb{B})$ is a nonempty convex compact subset of $\mathbb{P}(\mathbb{B})$.*

Proof. Since all the conditions of Theorem 3.3 are also assumed in the corollary, the conclusions of Theorem 3.3 hold. In particular, the game has the value. The additional conclusions follow from Theorem 2.7 (a) applied to the game $\{\mathbb{B}, \mathbb{A}, -c^{\mathbb{A}\leftrightarrow\mathbb{B}}\}$ introduced in Remark 2.13. ∎

The following assumption is symmetric to Assumption 2.8, and Corollary 3.6 from Theorem 3.3 is symmetric to Theorem 3.3.

Assumption 3.5. $\hat{c}^{\flat}(\pi^{\mathbb{B}}) < v^{\flat}$ for all $\pi^{\mathbb{B}} \in \mathbb{P}^{U}(\mathbb{B})$.

Corollary 3.6 (Existence of a value and solution for Player II). *Let a two-person zero-sum game $\{\mathbb{A}, \mathbb{B}, c\}$ satisfy Assumption 2.6 (b1,b2) and*

$$\hat{c}^{\flat}(\pi^{\mathbb{B}}) \leq v^{\flat} \ \text{for all} \ \pi^{\mathbb{B}} \in \mathbb{P}^{U}(\mathbb{B}). \tag{3.8}$$

Then the game $\{\mathbb{A}, \mathbb{B}, c\}$ has the value v. Moreover, if Assumption 3.5 holds, then the set $\mathbb{P}^{\flat}_{v}(\mathbb{B})$, which is a subset of $\mathbb{P}^{S}(\mathbb{B})$, is the set of solutions for Player II, and $\mathbb{P}^{\flat}_{v}(\mathbb{B})$ is a nonempty convex compact subset of $\mathbb{P}(\mathbb{B})$.

Proof. The corollary follows from Theorems 3.3 applied to the game $\{\mathbb{B}, \mathbb{A}, -c^{\mathbb{A}\leftrightarrow\mathbb{B}}\}$. ∎

The following statement

$$\hat{c}^{\flat}(\pi^{\mathbb{B}}) = -\infty \ \text{for all} \ \pi^{\mathbb{B}} \in \mathbb{P}^{U}(\mathbb{B}), \tag{3.9}$$

is similar and symmetric to statement (2.18). Note that statement (3.9) holds if and only if $\mathbb{P}^{\flat}_{\alpha}(\mathbb{B}) \subset \mathbb{P}^{S}(\mathbb{B})$ for each $\alpha \in \mathbb{R}$. The following theorem is similar and symmetric to Theorem 3.1.

Theorem 3.7 (Sufficient conditions for Assumption 3.5). *Assumption 3.5 holds if at least one of the following assumptions is satisfied:*

(C) *either assumption (L) or assumption (U) holds;*

(B1) *there exist $\gamma_{\mathbb{B}} \in (0, 1)$, $L_{\mathbb{B}} > 0$, and $a_0 \in \mathbb{A}$ such that for each $b \in \mathbb{B}$*

$$c^{-}(a_0, b) \leq \gamma_{\mathbb{B}} \hat{c}^{\flat}(b) + L_{\mathbb{B}}; \tag{3.10}$$

(B2) *there exist $\gamma_{\mathbb{B}} \in (0, 1)$, $L_{\mathbb{B}} > 0$, and $\pi^{\mathbb{A}}_0 \in \mathbb{P}(\mathbb{A})$ such that*

$$\int_{\mathbb{A}} \max\{0, \hat{c}^{\sharp}(a)\}\pi_0^{\mathbb{A}}(da) < +\infty, \quad and$$

$$\hat{c}^{\boxminus}(\pi_0^{\mathbb{A}}, b) \leq \gamma_{\mathbb{B}}\hat{c}^{\flat}(b) + L_{\mathbb{B}}, \qquad (3.11)$$

for each $b \in \mathbb{B}$;

(B3) *there exist* $\gamma_{\mathbb{B}} \in (0, 1)$ *and* $M_{\mathbb{B}} > 0$ *such that for each* $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$ *and* $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$

$$\hat{c}^{\flat}(\pi^{\mathbb{B}}) \leq \gamma_{\mathbb{B}}\hat{c}^{\boxminus}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) + M_{\mathbb{B}}; \qquad (3.12)$$

(B4) *there exists a function* $\Psi_{\mathbb{B}} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ *such that* $\Psi_{\mathbb{B}}(s) > -\infty$, *if* $s > -\infty$, *and for each* $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$ *and* $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$

$$\Psi_{\mathbb{B}}(\hat{c}^{\flat}(\pi^{\mathbb{B}})) \leq \hat{c}^{\boxminus}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}); \qquad (3.13)$$

(B5) *if* $\hat{c}^{\flat}(\pi^{\mathbb{B}}) > -\infty$ *for* $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$, *then* $\hat{c}^{\boxminus,\flat}(\pi^{\mathbb{B}}) > -\infty$.

Moreover, Assumption (B5) is equivalent to statement (3.9).

The proof of Theorem 3.7 is provided in Section 5.

Remark 3.8. Assumptions (C), (B1)–(B5) are useful for applications. A two-person zero-sum game $\{\mathbb{A}, \mathbb{B}, c\}$ satisfies Assumption (L) (Assumptions 3.5, (U), (C), (B1), (B2), (B3), (B4), (B5) respectively) if and only if the game $\{\mathbb{B}, \mathbb{A}, -c^{\mathbb{A}\leftrightarrow\mathbb{B}}\}$ introduced in Remark 2.13 satisfies Assumptions (U) (Assumptions 2.8, (L), (C), (A1), (A2), (A3), (A4), (A5) respectively).

The following theorem describes sufficient conditions for the existence of a solution for a game and the structure of the solution set.

Theorem 3.9 (Existence of a solution for a game). *Let a two-person zero-sum game* $\{\mathbb{A}, \mathbb{B}, c\}$ *satisfy Assumptions 2.6 (a1,a2,b1,b2) and 2.8, 3.5. Then:*

(i) *the game* $\{\mathbb{A}, \mathbb{B}, c\}$ *has a value* $v \in \mathbb{R}$ *and a solution* $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}_v^{\sharp}(\mathbb{A}) \times \mathbb{P}_v^{\flat}(\mathbb{B})$;

(ii) *the sets* $\mathbb{P}_v^{\sharp}(\mathbb{A})$ *and* $\mathbb{P}_v^{\flat}(\mathbb{B})$ *are nonempty convex compact subsets of* $\mathbb{P}(\mathbb{A})$ *and* $\mathbb{P}(\mathbb{B})$ *respectively; moreover,* $\mathbb{P}_v^{\sharp}(\mathbb{A}) \subset \mathbb{P}^S(\mathbb{A})$ *and* $\mathbb{P}_v^{\flat}(\mathbb{B}) \subset \mathbb{P}^S(\mathbb{B})$;

(iii) *a pair of strategies* $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}^S(\mathbb{A}) \times \mathbb{P}^S(\mathbb{B})$ *is a solution for the game* $\{\mathbb{A}, \mathbb{B}, c\}$ *if and only if* $\pi^{\mathbb{A}} \in \mathbb{P}_v^{\sharp}(\mathbb{A})$ *and* $\pi^{\mathbb{B}} \in \mathbb{P}_v^{\flat}(\mathbb{B})$.

The proof of Theorem 3.9 is provided in Section 5.

Remark 3.10. As explained in Remark 2.9, inf-compactness in a of the function $c(a, b)$ stated in Assumption 2.6 (a2) is close to the necessary condition for the existence of a solution for Player I. If the function c is bounded above on $\mathbb{A} \times \mathbb{B}$, then the set $\mathbb{A}$ is compact. The similar observation takes place for sup-compactness in b of the function $c(a, b)$ stated in Assumption 2.6

(b2), the existence of a solution for Player II, and compactness of the set $\mathbb{B}$. We observe that the expected payoff $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$ is well-defined for all pairs of strategies $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}(\mathbb{A}) \times \mathbb{P}(\mathbb{B})$ if and only if the function $c(a, b)$ is bounded either from above or from below on $\mathbb{A} \times \mathbb{B}$. In these two cases, the sets $\mathbb{A}$ and $\mathbb{B}$ are compact respectively. If the function c is bounded on $\mathbb{A} \times \mathbb{B}$, then under Assumption 2.6 (a2,b2) both decision sets are compact. So, for a problem with two noncompact decision sets $\mathbb{A}$ and $\mathbb{B}$, under natural inf/sup-compactness Assumption 2.6 (a2,b2) the payoff function is unbounded from above and from below on $\mathbb{A} \times \mathbb{B}$, and there exist pairs of strategies $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}(\mathbb{A}) \times \mathbb{P}(\mathbb{B})$ with undefined values of $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$.

If the function c is bounded above or below on $\mathbb{A} \times \mathbb{B}$, then Assumptions 2.8 and 3.5 hold; see Theorems 3.1 and 3.7. As follows from these observations and Remark 2.9, Theorems 3.3,3.9 and Corollary 3.4 imply several known results for games with two compact decision sets and with at least one compact decision set. In particular, Theorems 3.3 and Corollary 3.6 imply Glicksberg's theorem: for a game with two compact decision sets the value exists if the payoff function is upper (or lower) semi-continuous. If $\mathbb{A}$ and $\mathbb{B}$ are subsets of Polish spaces, then Theorem 3.9 implies Mertens et al. (2015, Theorem I.2.4) stating that the game has a solution if $\mathbb{A}$ and $\mathbb{B}$ are compact sets, c is a real-valued function bounded from below or from above, and Assumption 2.6 (a1,b1) hold (that is, for each $b \in \mathbb{B}$ the function $a \mapsto c(a, b)$ is lower semi-continuous, and for each $a \in \mathbb{A}$ the function $b \mapsto c(a, b)$ is upper semi-continuous). Indeed, if c is a real-valued function bounded from below or bounded from above, then, in view of Theorems 3.1 (L, U) and 3.7(C), Assumptions 2.8 and 3.5 hold, and, if the sets $\mathbb{A}$ and $\mathbb{B}$ are compact, then Assumption 2.6 (a1) implies Assumption 2.6 (a2), and Assumption 2.6 (b1) implies Assumption 2.6 (b2). We also remark that Mertens et al. (2015, Theorem I.2.4) is a more general statement that the version of von Neumann's theorem for mixed strategies, which states the existence of a solution, if the decision sets are compact and the payoff function is continuous; Owen (1982, Theorem IV.6.1) or Petrosyan and Zenkevich (2016, Theorem 2.4.4).

The following corollary from Theorem 3.9 generalizes Feinberg et al. (2022, Theorem 20).

Corollary 3.11 (Existence of a solution for a game). *Let a two-person zero-sum game* $\{\mathbb{A}, \mathbb{B}, c\}$ *satisfy Assumption 2.6 (a1,a2,b1,b2). If, in addition, the payoff function* c *is bounded either below or above on* $\mathbb{A} \times \mathbb{B}$, *then the conclusions of Theorem 3.9 hold.*

The proof of Corollary 3.11 is provided in Section 5.

As explained in Remark 2.9, if the function c is bounded above (below) in Corollary 3.11, then the set $\mathbb{A}$ ($\mathbb{B}$) is compact. The following example demonstrates that Assumption 2.6 (b2) is essential in Corollary 3.11 when the function c is bounded below on $\mathbb{A} \times \mathbb{B}$. Of course, this is also true for Assumption 2.6 (a2) when the function c is bounded above on $\mathbb{A} \times \mathbb{B}$.

Example 3.12. This example describes a two-person zero-sum game $\{\mathbb{A}, \mathbb{B}, c\}$ with the payoff function c bounded from below on $\mathbb{A} \times \mathbb{B}$ and satisfying Assumption 2.6 (a1,a2,b1). However, the function $b \mapsto c(a, b)$ is not sup-compact on $\mathbb{B}$ for each $a \in \mathbb{A}$, and $\mathbb{P}_v^\flat(\mathbb{B}) = \emptyset$. Therefore, this game has no solution.

Let $\mathbb{A} = \mathbb{B} := \mathbb{R}$, and

$$c(a, b) := 1 + a^2 - \frac{\exp(b)}{1 + \exp(b)}, \quad a, b \in \mathbb{R}.$$

Note that the function c takes positive values and it is continuous on $\mathbb{R}^2$. Moreover, the function $a \mapsto c(a, b)$ is obviously inf-compact on $\mathbb{R}$ for each $b \in \mathbb{R}$. However, for each $a \in \mathbb{R}$ the function $b \mapsto c(a, b)$ is not sup-compact on $\mathbb{R}$ because for every $a \in \mathbb{R}$ the set $\{b \in \mathbb{R} : c(a, b) \geq 0\} = \mathbb{R}$ is not compact.

The set $\mathbb{P}_v^\flat(\mathbb{B})$ is empty. Indeed, direct calculations imply that $v^\sharp = v^\flat = 0$ and $\hat{c}^\flat(\pi^\mathbb{B}) = 1 - \int_\mathbb{R} \frac{\exp(b)}{1+\exp(b)} \pi^\mathbb{B}(db)$ for each $\pi^\mathbb{B} \in \mathbb{P}(\mathbb{R})$. Therefore,

$$\mathbb{P}_v^\flat(\mathbb{B}) = \{\pi^\mathbb{B} \in \mathbb{P}(\mathbb{R}) : \int_\mathbb{R} \frac{\exp(b)}{1 + \exp(b)} \pi^\mathbb{B}(db) = 1\} = \emptyset,$$

where the last equality holds because $\pi^\mathbb{B}(\mathbb{R}) = 1$ and $\frac{\exp(b)}{1+\exp(b)} < 1$ for each $b \in \mathbb{R}$. The game $\{\mathbb{A}, \mathbb{B}, c\}$ has no solution since the set $\mathbb{P}_v^\flat(\mathbb{B})$ is empty.

4 | NUMBER GUESSING GAME

In this section, we consider the following game to illustrate the results of this article. Two players select nonnegative numbers a and b, and Player I pays the amount of $c(a, b) = \varphi(a - b)$ to Player II. For example, if the player, who selects the larger number wins, that is, $\varphi(a - b) = I(a > b)$, this game does not have a value; see for example, Yanovskaya (1974). We apply the results of our article to such games. In particular, for a polynomial function φ, Proposition 4.4 completely characterizes all the situations when the games have values and solutions.

Since both decision sets $\mathbb{A} = \mathbb{B} := \mathbb{R}_+ = [0, +\infty)$ are not compact, the only previously available result on the existence of the solution is Aubin and Ekeland (1984, Theorem 6.2.7), which assumes Assumption 2.6 (a1,a2), the concavity of $c(a, b)$ in a and convexity of $c(a, b)$ in b. Under these conditions, there exists a pure solution for the game. Another simple sufficient condition, under which a two-person zero-sum game $\{\mathbb{A}, \mathbb{B}, c\}$ has a pure solution, is $\mathbb{A} = \mathbb{B} := \mathbb{R}_+$ and the function $c(a, b)$ is nondecreasing in a and nonincreasing in b. In this case it is optimal for each player to select the decision 0. These arguments are applicable to Example 3.2.

The examples provided in this section may satisfy neither of the two described sufficient conditions. In addition, according to Proposition 4.4, solutions in pure strategies may not exist for the provided examples.

Example 4.1. Let $\mathbb{A} = \mathbb{B} := \mathbb{R}_+$ and $c(a, b) := \varphi(a - b)$ for each $a, b \in \mathbb{R}_+$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous function. To be consistent with assumptions (iv, v) in Definition 2.1, assume that

$$-\infty < \liminf_{s \to +\infty} \varphi(s) \quad \text{and} \quad \limsup_{s \to -\infty} \varphi(s) < +\infty. \quad (4.1)$$

The triple $\{\mathbb{A}, \mathbb{B}, c\}$ is a two-person zero-sum game introduced in Definition 2.1 because the function $a \mapsto c(a, b)$ is bounded below on $\mathbb{A}$ for each $b \in \mathbb{B}$, if the first inequality in (4.1) holds, and the function $b \mapsto c(a, b)$ is bounded above on $\mathbb{B}$ for each $a \in \mathbb{A}$, if the second inequality in (4.1) holds.

Proposition 4.2. *Consider the two-person zero-sum game defined in Example 4.1. Then:*

(a) *if $\varphi(s) \to +\infty$ as $s \to +\infty$, then Assumption 2.6 (a1,a2) and therefore the conclusions of Theorem 2.7 hold;*

(b) *if $\varphi(s) \to +\infty$ as $s \to +\infty$ and $\varphi(s) = \varphi_1(s) + \varphi_2(s)$ for each $s \in \mathbb{R}$, where $\varphi_1 : \mathbb{R} \to \mathbb{R}$ is increasing and $\varphi_2 : \mathbb{R} \to \mathbb{R}$ is bounded, then the assumptions and therefore the conclusions of Theorem 3.3 hold;*

(c) *if $\varphi(s) \to +\infty$ as $s \to +\infty$, $\varphi(s) \to -\infty$ as $s \to -\infty$, and $\varphi(s) = \varphi_1(s) + \varphi_2(s)$ for each $s \in \mathbb{R}$, where $\varphi_1 : \mathbb{R} \to \mathbb{R}$ is increasing and $\varphi_2 : \mathbb{R} \to \mathbb{R}$ is bounded, then the assumptions and therefore the conclusions of Theorem 3.9 hold.*

The proof of Proposition 4.2 is provided in Section 5.

Example 4.3. Consider Example 4.1 with the function φ being a polynomial of a degree $M = 1, 2, \ldots$, that is, $\varphi(s) = \sum_{n=0}^{M} \alpha_n s^n$, $s \in \mathbb{R}$, where $\alpha_n \in \mathbb{R}$, $n = 0, \ldots, M$, and $a_M \neq 0$.

Proposition 4.4. *Consider the two-person zero-sum game defined in Example 4.3.*

(a) *If the integer M is odd and $\alpha_M > 0$, then this game satisfies the assumptions of Theorem 3.9, and therefore the conclusions of Theorem 3.9 hold for this game.*

Furthermore, if $M \geq 3$ and $\alpha_1 < 0$, then there is no pure solution for this game.

(b) *If the integer M is even or $\alpha_M < 0$, then this game does not satisfy either assumption (iv) or assumption (v) from Definition 2.1, and there is no finite value because either $|v^\flat| = +\infty$ or $|v^\sharp| = +\infty$.*

The proof of Proposition 4.4 is provided in Section 5.

We note that in the two-person zero-sum game from Proposition 4.4 (a) action 0 for each Player strongly dominates any other action large enough and after elimination of these actions we have a compact game. On the other hand, let us consider the two-person zero-sum game defined in Example 4.1. If the assumptions of Proposition 4.2 (c) hold, and for each $s^* \in \mathbb{R} \setminus \{0\}$

$$\liminf_{s \to \pm\infty}(\varphi(s) - \varphi(s + s^*))$$
$$< 0 < \limsup_{s \to \pm\infty}(\varphi(s) - \varphi(s + s^*)), \quad (4.2)$$

then an any action of each player does not dominates any other his/her action because, according to (4.2), for each $a_*, a^* \in \mathbb{R}_+, a_* \neq a^*$, there exist $b_*, b^* \in \mathbb{R}_+$ such that

$$c(a_*, b_*) - c(a^*, b_*) = \varphi(a_* - b_*) - \varphi(a^* - b_*) < 0$$
$$< \varphi(a_* - b^*) - \varphi(a^* - b^*) = c(a_*, b^*) - c(a^*, b^*),$$

and, symmetrically, for each $b_*, b^* \in \mathbb{R}_+, b_* \neq b^*$, there exist $a_*, a^* \in \mathbb{R}_+$ such that

$$c(a^*, b_*) - c(a_*, b_*) = \varphi(a^* - b_*) - \varphi(a_* - b_*)$$
$$< 0 < \varphi(a^* - b_*) - \varphi(a_* - b^*) = c(a_*, b_*) - c(a_*, b^*).$$

The example of such function is $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1(s) = $ sgn $(s)\ln(|s| + 1)$ and $\varphi_2(s) = \sin(s) + \sin(\sqrt{2}s)$, $s \in \mathbb{R}$. Indeed, the assumptions of Proposition 4.2 (c) are trivial, inequalities (4.2) hold because $\varphi_1(s) - \varphi_1(s + s^*) \to 0$ as $s \to \pm\infty$, and φ_2 satisfies (4.2) for each $s^* \in \mathbb{R} \setminus \{0\}$ since the functions $s \mapsto \sin(s)$ and $s \mapsto (\sqrt{2}s)$ have commensurable prime periods.

We notice that Aubin and Ekeland (1984, Theorem 6.2.7) cannot be applied in most cases to the examples considered in this section because it assumes concavity of $c(a, b)$ in a and convexity of $c(a, b)$ in b. For example, assume that the function φ is twice differentiable, as this holds in Example 4.3. Then $\frac{\partial^2 \varphi(a-b)}{\partial a^2} = \frac{\partial^2 \varphi(a-b)}{\partial b^2}$. Therefore, the convexity/concavity assumption implies that these derivatives are equal to 0, and $\varphi(a - b) = M(a - b) + C$. For $M > 0$ this game is covered by Proposition 4.4(a), and $a = b = 0$ is the solution. For $M < 0$ this game is covered by Proposition 4.4 (ii), and there is no solution.

We also remark that the last claim of Proposition 4.4 (a), which states nonexistence of a pure solution, in some sense complements the result by Dreshen, Karlin, and Shapley (see Parrilo, 2006, Theorem 2.2) that states that, for a game with $\mathbb{A} = \mathbb{B} = [0, 1]$ and with a polynomial payoff function c, there exists a solution $(\pi^\mathbb{A}, \pi^\mathbb{B})$ with $\pi^\mathbb{A}$ and $\pi^\mathbb{B}$ having finite

supports. In the case of a polynomial function φ defined in Example 4.3, each of these finite supports consists of no more than $(M + 1)$ points.

5 | PROOFS

This section consists of four subsections. Section 5.1 provides the proofs of Theorems 3.1, 3.3, and 3.7, Section 5.2 provides the proofs of Theorem 3.9 and Corollary 3.11, and Section 5.3 provides the proofs of Propositions 4.2 and 4.4.

5.1 | Proofs of Theorems 3.1, 3.3, and 3.7

The proof of Theorem 3.1 is based on Lemmas 5.1–5.3.

Lemma 5.1. *Consider a two-person zero-sum game $\{\mathbb{A}, \mathbb{B}, c\}$. If $\pi^\mathbb{A} \in \mathbb{P}^S(\mathbb{A})$, then either $\hat{c}^{\boxplus,\sharp}(\pi^\mathbb{A}) < +\infty$ or $\inf_{b \in \mathbb{B}} \hat{c}^\boxminus(\pi^\mathbb{A}, b) > -\infty$.*

Proof. On the contrary, let $\hat{c}^{\boxplus,\sharp}(\pi^\mathbb{A}) = +\infty$ and $\inf_{b \in \mathbb{B}} \hat{c}^\boxminus(\pi^\mathbb{A}, b) = -\infty$ for a strategy $\pi^\mathbb{A} \in \mathbb{P}^S(\mathbb{A})$. Then for each $n = 1, 2, \dots$ there exist two points $b_n^{(1)}, b_n^{(2)} \in \mathbb{B}$ such that $\hat{c}^\boxplus(\pi^\mathbb{A}, b_n^{(1)}) \geq 2^n$ and $\hat{c}^\boxminus(\pi^\mathbb{A}, b_n^{(2)}) \leq -2^n$. Let us consider the probability measures $\pi_i^\mathbb{B}(B) = \sum_{n=1}^\infty 2^{-n} I\{b_n^{(i)} \in B\}$ for $B \in \mathcal{B}(\mathbb{B})$, $i = 1, 2$. We define $\pi^\mathbb{B} = \frac{1}{2}(\pi_1^\mathbb{B} + \pi_2^\mathbb{B})$. Then, by Fubini's theorem, $\hat{c}^\boxplus(\pi^\mathbb{A}, \pi_1^\mathbb{B}) = \sum_{n=1}^\infty \pi(b_n^{(1)})\hat{c}^\boxplus(\pi^\mathbb{A}, b_n^{(1)}) \geq \sum_{n=1}^\infty 2^{-n}2^n = +\infty$. Therefore, $\hat{c}^\boxplus(\pi^\mathbb{A}, \pi_1^\mathbb{B}) = +\infty$, and $\hat{c}^\boxplus(\pi^\mathbb{A}, \pi^\mathbb{B}) = \frac{1}{2}\hat{c}^\boxplus(\pi^\mathbb{A}, \pi_1^\mathbb{B}) + \frac{1}{2}\hat{c}^\boxplus(\pi^\mathbb{A}, \pi_2^\mathbb{B}) = +\infty$. Similarly, $\hat{c}^\boxminus(\pi^\mathbb{A}, \pi_2^\mathbb{B}) = -\infty$ which implies $\hat{c}^\boxminus(\pi^\mathbb{A}, \pi^\mathbb{B}) = -\infty$. Thus, $\pi^\mathbb{A} \notin \mathbb{P}^S(\mathbb{A})$. ∎

Lemma 5.2. *Assumption (A5) of Theorem 3.1 is equivalent to statement (2.18).*

Proof. Statement (2.18) is equivalent to its contrapositive statement

if $\hat{c}^\sharp(\pi^\mathbb{A}) < +\infty$ for $\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})$, then $\pi^\mathbb{A} \in \mathbb{P}^S(\mathbb{A})$. (5.1)

Let Assumption (A5) hold. Suppose $\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})$ satisfies the inequality $\hat{c}^\sharp(\pi^\mathbb{A}) < +\infty$. Observe that $\pi^\mathbb{A} \in \mathbb{P}^S(\mathbb{A})$. Indeed, since $\hat{c}^\sharp(\pi^\mathbb{A}) < +\infty$, then Assumption (A5) implies $\hat{c}^\boxplus(\pi^\mathbb{A}, \pi^\mathbb{B}) < +\infty$ for each $\pi^\mathbb{B} \in \mathbb{P}(\mathbb{B})$, that is, $\pi^\mathbb{A} \in \mathbb{P}^S(\mathbb{A})$. Thus, statement (5.1) holds.

Now let statement (5.1) hold. We consider an arbitrary $\pi^\mathbb{A} \in \mathbb{P}(\mathbb{A})$ satisfying the inequality $\hat{c}^\sharp(\pi^\mathbb{A}) < +\infty$. Then, in view of (5.1), $\pi^\mathbb{A} \in \mathbb{P}^S(\mathbb{A})$. Lemma 5.1 implies that either $\hat{c}^{\boxplus,\sharp}(\pi^\mathbb{A}) < +\infty$ or $\inf_{b \in \mathbb{B}} \hat{c}^\boxminus(\pi^\mathbb{A}, b) > -\infty$. To complete the proof of the validity of Assumption (A5), it is sufficient to show that the latter inequality implies the former one. Indeed, let $\inf_{b \in \mathbb{B}} \hat{c}^\boxminus(\pi^\mathbb{A}, b) >$

$-\infty$. Therefore, $\hat{c}^{\boxplus,\sharp}(\pi^{\mathbb{A}}) = \sup_{b\in\mathbb{B}}\{\hat{c}(\pi^{\mathbb{A}}, b) - \hat{c}^{\boxminus}(\pi^{\mathbb{A}}, b)\} \leq \hat{c}^{\sharp}(\pi^{\mathbb{A}}) - \inf_{b\in\mathbb{B}}\hat{c}^{\boxminus}(\pi^{\mathbb{A}}, b) < +\infty$. ∎

Lemma 5.3. *For a two-person zero-sum game* $\{\mathbb{A}, \mathbb{B}, c\}$, *the following implications hold for the assumptions introduced in Theorem 3.1:*

$$(U) \Rightarrow (A1) \Rightarrow (A2) \Rightarrow (A3) \Rightarrow (A4) \Rightarrow (A5). \quad (5.2)$$

Proof. "$(U) \Rightarrow (A1)$": If the function c is bounded above on $\mathbb{A} \times \mathbb{B}$, then inequality (3.3) takes place for $L_{\mathbb{A}} = \max\{0, \sup\{c(a,b) : a \in \mathbb{A}, b \in \mathbb{B}\}\} \in [0, +\infty)$, for all $\gamma_{\mathbb{A}} \in (0, 1)$, and for all $b_0 \in \mathbb{B}$.

"$(A1) \Rightarrow (A2)$": assumption (A1) implies assumption (A2) with $\pi_0^{\mathbb{B}} = \delta_{\{b_0\}}$ since (3.3) becomes the second inequality in (3.4), and the first one becomes $\hat{c}^{\flat}(b_0) > -\infty$, which is true in view of (2.3).

"$(A2) \Rightarrow (A3)$": Let us fix arbitrary $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$, $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$ and prove that

$$\gamma_{\mathbb{A}}\hat{c}^{\boxplus}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \leq \hat{c}^{\sharp}(\pi^{\mathbb{A}}) + L_{\mathbb{A}} - \int_{\mathbb{B}} \min\{0, \hat{c}^{\flat}(b)\}\pi_0^{\mathbb{B}}(db). \quad (5.3)$$

Note that

$$\sup_{b\in\mathbb{B}} c^+(a,b) = \max\{\hat{c}^{\sharp}(a), 0\}, \qquad a \in \mathbb{A}. \quad (5.4)$$

Indeed, if $\hat{c}^{\sharp}(a) \leq 0$, then $c(a,b) \leq 0$ for each $b \in \mathbb{B}$, and both sides of (5.4) equal 0. If $\hat{c}^{\sharp}(a) > 0$, then the set $B^+(a) := \{b \in \mathbb{B} : c(a,b) > 0\} = \{b \in \mathbb{B} : c^+(a,b) > 0\}$ is nonempty, and for each $a \in \mathbb{A}$

$$\sup_{b\in\mathbb{B}} c^+(a,b) = \sup_{b\in B^+(a)} c^+(a,b) = \sup_{b\in B^+(a)} c(a,b)$$
$$= \sup_{b\in\mathbb{B}} c(a,b) = \hat{c}^{\sharp}(a) = \max\{\hat{c}^{\sharp}(a), 0\};$$

these equalities follow from the basic properties of suprema and the definition of $B^+(a)$.

Equality (5.4) implies that for all $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$ and for all $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$

$$\hat{c}^{\boxplus}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) = \int_{\mathbb{A}}\int_{\mathbb{B}} c^+(a,b)\pi^{\mathbb{B}}(db)\pi^{\mathbb{A}}(da)$$
$$\leq \int_{\mathbb{A}}\int_{\mathbb{B}} \max\{\hat{c}^{\sharp}(a), 0\}\pi^{\mathbb{B}}(db)\pi^{\mathbb{A}}(da)$$
$$= \int_{\mathbb{A}} \max\{\hat{c}^{\sharp}(a), 0\}\pi^{\mathbb{A}}(da). \quad (5.5)$$

The second inequality in (3.4) implies

$$\gamma_{\mathbb{A}}\max\{\hat{c}^{\sharp}(a), 0\} \leq \hat{c}^{\boxplus}(a, \pi_0^{\mathbb{B}}) + L_{\mathbb{A}}, \qquad a \in \mathbb{A}. \quad (5.6)$$

The integration of both sides of (5.6) in $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$ leads to

$$\gamma_{\mathbb{A}}\int_{\mathbb{A}} \max\{\hat{c}^{\sharp}(a), 0\}\pi^{\mathbb{A}}(da) \leq \hat{c}^{\boxplus}(\pi^{\mathbb{A}}, \pi_0^{\mathbb{B}}) + L_{\mathbb{A}}$$
$$= \hat{c}(\pi^{\mathbb{A}}, \pi_0^{\mathbb{B}}) - \hat{c}^{\boxminus}(\pi^{\mathbb{A}}, \pi_0^{\mathbb{B}}) + L_{\mathbb{A}}. \quad (5.7)$$

Observe that

$$-\infty < \int_{\mathbb{B}} \min\{0, \hat{c}^{\flat}(b)\}\pi_0^{\mathbb{B}}(db)$$
$$= \int_{\mathbb{A}}\int_{\mathbb{B}} \min\{0, \hat{c}^{\flat}(b)\}\pi_0^{\mathbb{B}}(db)\pi^{\mathbb{A}}(da) \leq \hat{c}^{\boxminus}(\pi^{\mathbb{A}}, \pi_0^{\mathbb{B}}), \quad (5.8)$$

where the first inequality in (5.8) is the first inequality in (3.4), the equality follows from integrating the constant in $\pi^{\mathbb{A}}$, and the last inequality follows from the second inequality in (2.3).

Inequalities (5.7) and (5.8) imply that

$$\gamma_{\mathbb{A}}\int_{\mathbb{A}} \max\{\hat{c}^{\sharp}(a), 0\}\pi^{\mathbb{A}}(da) \leq \hat{c}(\pi^{\mathbb{A}}, \pi_0^{\mathbb{B}})$$
$$+ L_{\mathbb{A}} - \int_{\mathbb{B}} \min\{0, \hat{c}^{\flat}(b)\}\pi_0^{\mathbb{B}}(db). \quad (5.9)$$

Observe that $-\infty < \hat{c}(\pi^{\mathbb{A}}, \pi_0^{\mathbb{B}}) \leq \hat{c}^{\sharp}(\pi^{\mathbb{A}})$ because of (5.8) and the definition of $\hat{c}^{\sharp}(\pi^{\mathbb{A}})$. Therefore, inequalities (5.5) and (5.9) imply (5.3).

Let us choose $\gamma_{\mathbb{A}} \in (0, 1)$ and $M_{\mathbb{A}} := \max\{L_{\mathbb{A}} - \int_{\mathbb{B}} \min\{0, \hat{c}^{\flat}(b)\}\pi_0^{\mathbb{B}}(db), 1\} > 0$. Note that the first inequality in (3.4) implies $M_{\mathbb{A}} < +\infty$. Thus (3.5) follows from (5.3).

"$(A3) \Rightarrow (A4)$": Inequality (3.6) follows from (3.5) if we set $\Psi_{\mathbb{A}}(s) := \frac{1}{\gamma_{\mathbb{A}}}(s + M_{\mathbb{A}})$ for each $s \in \overline{\mathbb{R}}$.

"$(A4) \Rightarrow (A5)$": Inequality (3.6) implies that $\hat{c}^{\boxplus,\sharp}(\pi^{\mathbb{A}}) \leq \Psi_{\mathbb{A}}(\hat{c}^{\sharp}(\pi^{\mathbb{A}})) < +\infty$ if $\hat{c}^{\sharp}(\pi^{\mathbb{A}}) < +\infty$. ∎

Proof of Theorem 3.1. The equivalence statement follows from Lemma 5.2.

"$(L) \Rightarrow (A3)$": assumption (L), stating boundedness below of the function c, means that there exists a real number $\gamma \geq 0$ such that $c(x, a) \geq -\gamma > -\infty$ for all $(a, b) \in \mathbb{A} \times \mathbb{B}$. This inequality implies that $c(a, b) + \gamma \geq c^+(a, b)$ for all $(a, b) \in \mathbb{A} \times \mathbb{B}$. By integrating both sides of the last inequality in $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$ and taking the supremum in $b \in \mathbb{B}$, we have $\hat{c}^{\sharp}(\pi^{\mathbb{A}}) + \gamma \geq \hat{c}^{\boxplus,\sharp}(\pi^{\mathbb{A}})$ for all $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$. Therefore, for an arbitrary fixed $\gamma_{\mathbb{A}} \in (0, 1)$ and $M_{\mathbb{A}} := \gamma$ we have that $\gamma_{\mathbb{A}}\hat{c}^{\boxplus}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \leq \hat{c}^{\sharp}(\pi^{\mathbb{A}}) + M_{\mathbb{A}}$ for each $\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})$ and $\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})$, that is, assumption (A3) holds.

The remaining statements of Theorem 3.1 follow from Lemma 5.3. ∎

Proof of Theorem 3.3. Condition (3.7) directly implies formula (2.16). In view of Theorem 2.7, the value of the game v exists and the set $\mathbb{P}_v^{\sharp}(\mathbb{A})$ is nonempty and convex. Therefore, according to Assumption 2.8 and (2.17), $\mathbb{P}_v^{\sharp}(\mathbb{A}) \subset \mathbb{P}^S(\mathbb{A})$. As follows from Definitions 2.3 and 2.4, $\mathbb{P}_v^{\sharp}(\mathbb{A})$ is the set of solutions for Player I. ∎

The following corollary follows from Lemma 5.3.

Corollary 5.4. *For a two-person zero-sum game* $\{\mathbb{A}, \mathbb{B}, c\}$,

$$(L) \Rightarrow (B1) \Rightarrow (B2) \Rightarrow (B3) \Rightarrow (B4) \Rightarrow (B5). \quad (5.10)$$

Proofs of Theorem 3.7 and Corollary 5.4. According to Remark 3.8, Theorem 3.7, and Corollary 5.4 follow respectively from Theorem 3.1 and Lemma 5.3 applied to the game $\{\mathbb{B}, \mathbb{A}, -c^{\mathbb{A}\leftrightarrow\mathbb{B}}\}$ introduced in Remark 2.13. ∎

5.2 | Proofs of Theorem 3.9 and Corollary 3.11

Proof of Theorem 3.9. Theorem 3.3 states the existence of the value. It also states that $\mathbb{P}_\nu^\sharp(\mathbb{A})$ is the set of the solutions for Player I, $\mathbb{P}_\nu^\sharp(\mathbb{A}) \subset \mathbb{P}^S(\mathbb{A})$, and $\mathbb{P}_\nu^\sharp(\mathbb{A})$ is a nonempty convex compact subset of $\mathbb{P}(\mathbb{A})$. Corollary 3.6 states that $\mathbb{P}_\nu^\flat(\mathbb{B})$ is the set of the solutions for Player II, $\mathbb{P}_\nu^\flat(\mathbb{B}) \subset \mathbb{P}^S(\mathbb{B})$, and $\mathbb{P}_\nu^\flat(\mathbb{B})$ is a nonempty convex compact subset of $\mathbb{P}(\mathbb{B})$. As explained in the paragraph following formula (2.22), $\mathbb{P}_\nu^\sharp(\mathbb{A}) \times \mathbb{P}_\nu^\flat(\mathbb{B})$ is the set of all solutions for the game. ∎

Proof of Corollary 3.11. The corollary directly follows from Theorems 3.1, 3.7, and 3.9. Indeed, Theorems 3.1 and 3.7 imply that Assumptions 2.8 and 3.5 hold. Therefore, all assumptions of Theorem 3.9 hold because, in addition, the game $\{\mathbb{A}, \mathbb{B}, c\}$ satisfy Assumption 2.6 (a1,a2,b1,b2). ∎

5.3 | Proofs of Propositions 4.2 and 4.4

We start this subsection with some definitions and auxiliary lemmas. We recall that for metric spaces $\mathbb{X}$ and $\mathbb{Y}$ a function $f : \mathbb{X} \times \mathbb{Y} \to \overline{\mathbb{R}}$ is called $\mathbb{K}$-*inf-compact on* $\mathbb{X} \times \mathbb{Y}$, if for every compact set $K \subset \mathbb{X}$ this function is inf-compact on $K \times \mathbb{Y}$; see Feinberg et al. (2013, Definition 1.1, 2022, Definition 1). A function $f : \mathbb{X} \times \mathbb{Y} \to \overline{\mathbb{R}}$ is called $\mathbb{K}$-*sup-compact on* $\mathbb{X} \times \mathbb{Y}$, if the function $-f$ is $\mathbb{K}$-inf-compact on $\mathbb{X} \times \mathbb{Y}$; see Feinberg et al. (2022, Definition 2). We would like to clarify that in this article we consider $\mathbb{K}$-inf-compactness and $\mathbb{K}$-sup-compactness on the set $\mathbb{X} \times \mathbb{Y}$, which the the graph of the set-value mapping $\Phi : \mathbb{X} \to 2^{\mathbb{Y}}$ with $\Phi(x) = \mathbb{Y}$ for all $x \in \mathbb{X}$, while in Feinberg et al. (2013, 2022) these definitions were considered for the graph of an arbitrary multifunction Φ, where $\Phi : \mathbb{X} \to 2^{\mathbb{Y}} \setminus \{\emptyset\}$ in Feinberg et al. (2013) and $\mathbb{X} : \Phi \to 2^{\mathbb{Y}}$ in Feinberg et al. (2022). We observe that, if a function $f : \mathbb{X} \times \mathbb{Y} \to \overline{\mathbb{R}}$ is $\mathbb{K}$-inf-compact ($\mathbb{K}$-sup-compact) on $\mathbb{X} \times \mathbb{Y}$, then for each $x \in \mathbb{X}$ the function $y \mapsto f(x, y)$ is inf-compact on $\mathbb{Y}$. This follows from the observation that every singleton $K = \{x\}$, $x \in \mathbb{X}$, is compact.

Lemma 5.5 (Feinberg et al., 2022). *The function* $f : \mathbb{X} \times \mathbb{Y} \to \overline{\mathbb{R}}$ *is* $\mathbb{K}$-*inf-compact on* $\mathbb{X} \times \mathbb{Y}$ *if and only if the following two assumptions hold:*

(i) $f : \mathbb{X} \times \mathbb{Y} \to \overline{\mathbb{R}}$ *is lower semi-continuous;*
(ii) *if a sequence* $\{x^{(n)}\}_{n=1,2,\dots}$ *with values in* $\mathbb{X}$ *converges in* $\mathbb{X}$ *and its limit* x *belongs to* $\mathbb{X}$, *then each sequence* $\{y^{(n)}\}_{n=1,2,\dots} \subset \mathbb{Y}$ *satisfying the condition that the sequence* $\{f(x^{(n)}, y^{(n)})\}_{n=1,2,\dots}$ *is bounded above, has a limit point* $y \in \mathbb{Y}$.

The following lemma is the main technical fact in this subsection.

Lemma 5.6. *Let* $\mathbb{A} = \mathbb{B} := \mathbb{R}_+$ *and* $c(a, b) := \varphi(a - b)$ *for each* $a, b \in \mathbb{R}_+$, *where* $\varphi : \mathbb{R} \to \mathbb{R}$ *is a continuous function. Then the following statements hold:*

(i) *if* $\varphi(s) \to +\infty$ *as* $s \to +\infty$, *then the function* $(b, a) \mapsto c(a, b)$ *is* $\mathbb{K}$-*inf-compact on* $\mathbb{B} \times \mathbb{A}$;
(ii) *if* $\varphi(s) \to -\infty$ *as* $s \to -\infty$, *then the function* $(a, b) \mapsto c(a, b)$ *is* $\mathbb{K}$-*sup-compact on* $\mathbb{A} \times \mathbb{B}$;
(iii) *if* $\varphi(s) = \varphi_1(s) + \varphi_2(s)$ *for each* $s \in \mathbb{R}$, *where* $\varphi_1 : \mathbb{R} \to \mathbb{R}$ *is increasing and* $\varphi_2 : \mathbb{R} \to \mathbb{R}$ *is bounded, then assumptions (A1) and (B1) from Theorems 3.1 and 3.7 hold;*
(iv) *if there exist* $s_* < 0 < s^*$ *such that* $\varphi(s_*) > \varphi(s^*)$, *then the game* $\{\mathbb{A}, \mathbb{B}, c\}$ *has no pure solution.*

Proof.

(i) We verify the conditions of Lemma 5.5 to prove $\mathbb{K}$-inf-compactness of the function $c(a, b) = \varphi(a - b)$. This function is continuous, and therefore it is lower semi-continuous. Consider a sequence $\{b^{(n)}\}_{n\geq1}$ that converges to $b \in \mathbb{B}$ and a sequence $\{a^{(n)}\}_{n\geq1} \subset \mathbb{A}$ such that $\{\varphi(a^{(n)} - b^{(n)})\}_{n\geq1}$ is bounded above. Since the sequence $\{b^{(n)}\}_{n\geq1} \subset \mathbb{R}_+$ converges, it is bounded. Since the sequence $\{\varphi(a^{(n)} - b^{(n)})\}_{n\geq1}$ is bounded above, then the continuity of the function $\varphi : \mathbb{R} \to \mathbb{R}$ on $\mathbb{R}$ and the property $\varphi(s) \to +\infty$ as $s \to +\infty$ imply that the sequence $\{a^{(n)} - b^{(n)}\}_{n\geq1}$ is bounded above. Thus, the sequence $\{a^{(n)}\}_{n\geq1} \subset \mathbb{R}_+$ is bounded above and therefore it is bounded. Therefore, the sequence $\{a^{(n)}\}_{n\geq1}$ has an accumulation point $a \in \mathbb{A}$. Thus, the assumptions of Lemma 5.5 are verified, and the function c is $\mathbb{K}$-inf-compact.
(ii) This statement follows from (i) applied to the game $\{\mathbb{B}, \mathbb{A}, -c^{\mathbb{A}\leftrightarrow\mathbb{B}}\}$.

(iii) First, we prove that assumption (B1) holds. Let the function φ be the sum of the functions φ_1 and φ_2 described in the statement. Then for each $b \geq 0$

$$
\begin{aligned}
c^{\flat}(b) &= \inf_{a \geq 0} \{\varphi_1(a - b) + \varphi_2(a - b)\} \\
&\geq \inf_{a \geq 0} \varphi_1(a - b) + \inf_{a \geq 0} \varphi_2(a - b) \\
&= \varphi_1(-b) + \inf_{a \geq 0} \varphi_2(a - b) = c(0, b) \\
&\quad + \inf_{a \geq 0} \varphi_2(a - b) \geq c(0, b) - B,
\end{aligned}
$$

where $B > 0$ is a constant such that $|\varphi_2(s)| \leq B$ for each $s \in \mathbb{R}$. We note that the second equality holds because the function φ_1 is increasing. Therefore, for each $b \geq 0$

$$
c^-(0, b) \leq \frac{1}{2} c^-(0, b) \leq \frac{1}{2} c^{\flat}(b) + \frac{B}{2}.
$$

This implies that assumption (B1) holds. Second, assumption (A1) holds because it is equivalent to assumption (B1) for the game $\{\mathbb{B}, \mathbb{A}, -c^{\mathbb{A} \leftrightarrow \mathbb{B}}\}$, which holds because the real function φ_1 is increasing if and only if the real function $s \mapsto -\varphi_1(-s)$ is increasing, and the function φ_2 is bounded if and only if the function $s \mapsto -\varphi_2(-s)$ is bounded.

(iv) There exist $s_*, s^* \in \mathbb{R}$ such that $s_* < 0 < s^*$ and $\varphi(s_*) > \varphi(s^*)$. Then for each $a, b \geq 0$

$$
\begin{aligned}
c^{\flat}(b) &= \inf_{a^* \geq 0} \varphi(a^* - b) \leq \varphi(s^*), \\
c^{\sharp}(a) &= \sup_{b^* \geq 0} \varphi(a - b^*) \geq \varphi(s_*).
\end{aligned}
$$

Therefore,

$$
\sup_{b \geq 0} c^{\flat}(b) \leq \varphi(s^*) < \varphi(s_*) \leq \inf_{a \geq 0} c^{\sharp}(a),
$$

that is, the game $\{\mathbb{A}, \mathbb{B}, c\}$ has no pure solution. ∎

Proof of Proposition 4.2.. (a) In view of Lemma 5.6 (i), the function $(b, a) \mapsto c(a, b)$ is $\mathbb{K}$-inf-compact on $\mathbb{B} \times \mathbb{A}$. This implies Assumption 2.6 (a1,a2). Statement (b) follows from Lemma 5.6 (i,iii). Statement (c) follows from Lemma 5.6 (i-iii). ∎

Proof of Proposition 4.4. (a) This statement follows from Proposition 4.2 (a) and Lemma 5.6 (iv).

(b) Let us consider three cases (c1–c3).

(c1) Let M be even and $\alpha_M > 0$. Then condition (v) from Definition 2.1 does not hold because the function $b \mapsto c(a, b)$ is not bounded above on $\mathbb{R}$ for each $a \geq 0$. Since the function φ is bounded below on $\mathbb{R}$, the value $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$ is well-defined

for all $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}(\mathbb{A}) \times \mathbb{P}(\mathbb{B})$ and

$$
\sup_{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) = \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \sup_{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) = +\infty.
$$

$$(5.11)$$

Indeed, if we set $\pi^{\mathbb{B}}(B) := \frac{2}{\pi} \int_B \frac{1}{1 + b^2} db$ for each $B \in \mathcal{B}(\mathbb{B})$, then for all $a \in \mathbb{A}$

$$
\hat{c}(a, \pi^{\mathbb{B}}) = \frac{2}{\pi} \int_{\mathbb{R}_+} \frac{\varphi(a - b)}{1 + b^2} db = +\infty.
$$

Therefore, $\hat{c}^{\flat}(\pi^{\mathbb{B}}) = \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) = \inf_{a \geq 0} \hat{c}(a, \pi^{\mathbb{B}}) = +\infty$, and

$$
\begin{aligned}
+\infty &\leq \hat{c}^{\flat}(\pi^{\mathbb{B}}) \leq \sup_{\pi_*^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \hat{c}^{\flat}(\pi_*^{\mathbb{B}}) = \sup_{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \\
&\leq \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \sup_{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}).
\end{aligned}
$$

Thus equalities (5.11) hold.

(c2) If M is even and $\alpha_M < 0$, then condition (iv) from Definition 2.1 does not hold because the function $a \mapsto c(a, b)$ is not bounded below on $\mathbb{R}$ for each $b \geq 0$. Since the function φ is bounded above on $\mathbb{R}$, the value $\hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}})$ is well-defined for all $(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) \in \mathbb{P}(\mathbb{A}) \times \mathbb{P}(\mathbb{B})$. Moreover, by the symmetric reasonings, which follow from case (c1),

$$
\sup_{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) = \inf_{\pi^{\mathbb{A}} \in \mathbb{P}(\mathbb{A})} \sup_{\pi^{\mathbb{B}} \in \mathbb{P}(\mathbb{B})} \hat{c}(\pi^{\mathbb{A}}, \pi^{\mathbb{B}}) = -\infty.
$$

(c3) If M is odd and $\alpha_M < 0$, then conditions (iv,v) from Definition 2.1 do not hold. Moreover, the lower value for this game in pure strategies equals $-\infty$, and the upper value for this game in pure strategies equals $+\infty$. ∎

ACKNOWLEDGMENTS

DATA AVAILABILITY STATEMENT

This article does not use data.

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