

HARMONIC ANALYSIS ON A LOCALLY COMPACT HYPERGROUP

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ABSTRACT. We propose a new axiomatics for a locally compact hypergroup. On the one hand, the new object generalizes a DJS-hypergroup and, on the other hand, it allows to obtain results similar to those for a unimodular hypercomplex system with continuous basis. We construct a harmonic analysis and, for a commutative locally compact hypergroup, give an analogue of the Pontryagin duality theorem.

1. INTRODUCTION

An extensive study of generalized translation operators or, in other terms, a coalgebra structure on a function algebra with a right counit has started in the work of J. Delsart [1] and B. Levitan [2], also see the more contemporary surveys of B. Litvinov [3] and L. Vainerman [4]. Yu. M. Berezansky and S. G. Krein [5, 6] have introduced a notion of a commutative hypercomplex system with a continual (compact or discrete) basis, where, for the first time, apparently, an axiom of positivity of comultiplication has been introduced, which gave a possibility to develop a rich harmonic analysis for such systems. Starting in 1973, after a DJS-hypergroup was introduced independently by C. Dunkl [7], R. Spector [8], and R. Jewett [9], there appeared a significant number of works on the DJS-hypergroups, see e.g. the monograph [10] and the bibliography therein. It turned out that compact and discrete commutative DJS-hypergroups make a subclass of hypercomplex systems with compact and discrete bases, hence a number of the results on such hypergroups have already been obtained in earlier works of Yu. M. Berezansky and S. G. Krein. Moreover, the definition of a DJS-hypergroup imposes certain topological type conditions that are not used in the definition of a hypercomplex system of Yu. M. Berezansky and S. G. Krein; also, these conditions do not yield a satisfactory analogue of the Pontryagin duality in the case of a commutative DJS-hypergroup.

Unimodular locally compact hypercomplex systems, noncommutative in general, have been studied by Yu. M. Berezansky and A. A. Kalyuzhnyi, see [11] and the bibliography cited there. Such hypercomplex systems are more general than the unimodular locally compact DJS-hypergroups, although the axiomatics of such hypercomplex systems is rather cumbersome and can not be generalized to a nonunimodular case.

For this reason, in this paper, we introduce a locally compact hypergroup which generalize a DJS-hypergroup, on the one hand, and generalize a normal hypercomplex system with a basis unity to the nonunimodular case, on the other hand.

The definition of the locally compact hypergroup is given in terms of a comultiplication, which is the same as defining it via generalized translation operators (recall that the comultiplication Δ and the generalized translation operators R_p are connected by the formula $(R_p f)(q) = (\Delta f)(p, q)$, where $p, q \in Q$, f is a continuous bounded function on the hypergroup Q). Moreover, as it is the case with locally compact quantum groups [12],

2000 *Mathematics Subject Classification.* Primary: 20N20; Secondary: 22D15, 22D35.

Key words and phrases. Hypergroup, DJS-hypergroup, locally compact hypergroup, hypercomplex system, generalized translation operator, dual algebras, Pontryagin duality.

This work has been partially supported by the Scientific Program of the National Academy of Sciences of Ukraine, Project No 0107U002333.

the axiomatics of the locally compact hypergroup includes an axiom for existence of a left Haar measure. This is justified, since existence of a left Haar measure has not been proved even for a locally compact DJS-hypergroup (its existence was proved only for a discrete, a compact, or a commutative DJS-hypergroup). On the other hand, postulating existence of a Haar measure allows to simplify the axiomatics and to extend the class of examples of locally compact hypergroups; there exists an example of a compact commutative hypergroup related to the generalized Tchebycheff polynomials, which is not a DJS-hypergroup [11].

Let us now describe the structure of the paper.

Section 2 gives a definition of a locally compact hypergroup and examples related to groups and double cosets of a group and a subgroup.

In Section 3, we study convolution of measures. Let us mention that the convolution of Dirac measures is given by the formula $(\epsilon_p * \epsilon_q)(f) = (\Delta f)(p, q)$, where $p, q \in Q$ and f is a continuous bounded function of the hypergroup Q .

In Section 4 we prove that the left and the right Haar measures are unique up to a scalar. We also introduce a corresponding modulus function.

In Section 5, we prove that the space $L_1(Q, \mu)$ of functions on the hypergroup Q , which are absolutely integrable with respect to a left Haar measure μ , is an involutive algebra with an approximate identity, with respect to naturally defined convolution and involution.

It is shown in Section 6 that the space of continuous functions with compact support form a left Hilbert algebra with respect to the corresponding convolution. We also study modular properties of the obtained left Hilbert algebra.

Regular representations of the locally compact hypergroup are introduced in Section 7, where we prove a theorem on a correspondence between bounded representation of the hypergroup and \star -representations of the involutive algebra $L_1(Q, \mu)$.

Section 8 gives a study of positive definite functions on the locally compact hypergroup, and contains a theorem on approximating, with respect to the topology of uniform convergence on compact sets, a continuous function on the hypergroup with linear combinations of elementary positive definite functions.

In section 9, we prove that the von Neumann algebras generated by the left and right regular representations are commutants of each other, which is similar to the group case. Also we prove an analogue of the Plancherel theorem and an inversion formula.

Section 10 gives main theorems on harmonic analysis for commutative hypergroups, including an analogue of the Pontryagin duality theorem.

2. DEFINITION OF A LOCALLY COMPACT HYPERGROUP. EXAMPLES.

Let Q be a locally compact second countable Hausdorff topological space.

The spaces of complex-valued functions that are continuous, continuous and bounded, continuous with compact supports, continuous and equal to zero at infinity are denoted by $\mathcal{C}(Q)$, $\mathcal{C}_b(Q)$, $\mathcal{C}_c(Q)$, $\mathcal{C}_0(Q)$, respectively. For $f \in \mathcal{C}_c(Q)$, $\text{supp } f$ denotes support of the function f . The linear spaces $\mathcal{C}_0(Q)$ and $\mathcal{C}_b(Q)$ have the structure of a C^* -algebra with respect to the pointwise multiplication and complex conjugation, endowed with the norm $\|f\| = \sup_{r \in Q} |f(r)|$ for $f \in \mathcal{C}_b(Q)$ or $f \in \mathcal{C}_0(Q)$.

Everywhere in the sequel a *measure* will mean a Radon measure on Q [13]. The integral of f , $f \in \mathcal{C}_c(Q)$, with respect to a measure μ is denoted by $\mu(f) = \int_Q f(p) d\mu(p)$. The Dirac measure at a point $q \in Q$ is denoted by ϵ_q , i.e., $\epsilon_q(f) = f(q)$, $f \in \mathcal{C}(Q)$. For a measure μ , its *absolute value* $|\mu|$ and the *norm* $\|\mu\|$ are, respectively, $|\mu|(f) = \sup_{g \in \mathcal{C}_c(Q), |g| \leq f} |\mu(g)|$ for $f \geq 0$, and $\|\mu\| = \sup_{f \in \mathcal{C}_c(Q), \|f\| \leq 1} |\mu(f)|$. We use $\mathcal{M}(Q)$, $\mathcal{M}_b(Q)$, $\mathcal{M}_c(Q)$ to denote, respectively, the set of *measures* on Q , the set of *bounded*

measures, the set of measures with compact supports. The linear space $\mathcal{M}_b(Q)$ is a Banach space with respect to the norm $\|\cdot\|$ [13].

Definition 2.1. Let Q be a locally compact space with an involutive homeomorphism $*$: $Q \rightarrow Q$ and a point $e \in Q$, $e^* = e$, and let the following conditions be satisfied.

(H_1) There is a \mathbb{C} -linear mapping $\Delta: \mathcal{C}_b(Q) \rightarrow \mathcal{C}_b(Q \times Q)$ such that
(a) Δ is coassociative, that is,

$$(1) \quad (\Delta \times \text{id}) \circ \Delta = (\text{id} \times \Delta) \circ \Delta;$$

(b) Δ is positive, that is, $\Delta f \geq 0$ for all $f \in \mathcal{C}_b(Q)$ such that $f \geq 0$;

(c) Δ preserves the identity, that is, $(\Delta 1)(p, q) = 1$, for all $p, q \in Q$;

(d) for all $f, g \in \mathcal{C}_c(Q)$, we have $(1 \otimes f) \cdot (\Delta g) \in \mathcal{C}_c(Q \times Q)$ and $(f \otimes 1) \cdot (\Delta g) \in \mathcal{C}_c(Q \times Q)$.

(H_2) The homomorphism $\epsilon: \mathcal{C}_b(Q) \rightarrow \mathbb{C}$ defined on the C^* -algebra $\mathcal{C}_b(Q)$ by $\epsilon(f) = f(e)$ satisfies the counit property, that is,

$$(2) \quad (\epsilon \times \text{id}) \circ \Delta = (\text{id} \times \epsilon) \circ \Delta = \text{id},$$

in other words, $(\Delta f)(e, p) = (\Delta f)(p, e) = f(p)$ for all $p \in Q$.

(H_3) The function \check{f} defined by $\check{f}(q) = f(q^*)$ for $f \in \mathcal{C}_b(Q)$ satisfies

$$(3) \quad (\Delta \check{f})(p, q) = (\Delta f)(q^*, p^*).$$

(H_4) There exists a positive measure μ on Q , $\text{supp } \mu = Q$, such that

$$(4) \quad \int_Q (\Delta f)(p, q) g(q) d\mu(q) = \int_Q f(q) (\Delta g)(p^*, q) d\mu(q)$$

for all $f \in \mathcal{C}_b(Q)$ and $g \in \mathcal{C}_c(Q)$, or $f \in \mathcal{C}_c(Q)$ and $g \in \mathcal{C}_b(Q)$, $p \in Q$; such a measure μ will be called a *left Haar measure* on Q .

Then $(Q, *, e, \Delta, \mu)$, or simply Q , is called a *locally compact hypergroup*.

Definition 2.2. Let Q be a locally compact hypergroup. A positive measure ν on Q satisfying

$$(5) \quad \int_Q (\Delta f)(p, q) g(p) d\nu(p) = \int_Q f(p) (\Delta g)(p, q^*) d\nu(p)$$

for all f, g as in (H_4), $q \in Q$, is called a *right Haar measure* on Q .

Remark 2.3. It directly follows from axiom (H_1) (d) that the integrands in (4) (resp., (5)) are compact functions of q (resp., p) for a fixed p (resp., q), hence the integrals in (4) and (5) are finite.

Example 2.4. Let Q be a locally compact group with multiplication \cdot , unit e , and inverse $^{-1}$. For $f \in \mathcal{C}_b(Q)$, $p, q \in Q$, define

$$(\Delta f)(p, q) = f(p \cdot q),$$

take e to be the unit of Q , and $q^* = q^{-1}$, $q \in Q$, μ the left Haar measure on Q . Then Q becomes a locally compact hypergroup.

Axiom (H_1) (a) asserts that the multiplication in the group is associative, $f((p \cdot q) \cdot r) = f(p \cdot (q \cdot r))$ for all $f \in \mathcal{C}_b(Q)$. Axiom (H_1) (b), (H_1) (c) are clear. Axiom (H_1) (d) requires that $f(p)g(p \cdot q)$ have compact support for $f, g \in \mathcal{C}_c(Q)$, which is true. Since e is a unit in Q , we have $(\epsilon \times \text{id}) \circ \Delta(f)(p) = f(e \cdot p) = f(p)$ and $(\text{id} \times \epsilon) \circ \Delta(f)(p) = f(p \cdot e) = f(p)$ showing that (H_2) is verified. (H_3) means that $f((p \cdot q)^{-1}) = f(q^{-1} \cdot p^{-1})$, and, finally, condition (H_4) means that

$$\int_Q f(p \cdot q) g(q) d\mu(q) = \int_Q f(q) g(p^{-1} \cdot q) d\mu(q),$$

which is equivalent to that μ is indeed a left Haar measure on the group Q .

Note that Δ is a C^* -algebra homomorphism here, which needs not be the case in general.

Example 2.5. Let G be a locally compact group with multiplication \cdot , unit e , and inverse $^{-1}$, and let μ_G be a left Haar measure on G . Let H be a compact subgroup of G with a Haar measure μ_H normalized by the condition $\int_H d\mu_H(p) = 1$. Let $Q = H \backslash G / H = \{HgH : g \in G\}$ be the set of double cosets endowed with the factor topology. Define $P: \mathcal{C}_b(G) \rightarrow \mathcal{C}_b(Q)$ by

$$P(f)(g) = \int_{H^2} f(h_1 g h_2) d\mu_H(h_1) d\mu_H(h_2).$$

Then $\mathcal{C}_b(Q) \cong P(\mathcal{C}_b(G))$ and $\mathcal{C}_0(Q) \cong P(\mathcal{C}_0(G))$, and it is easy to see that $f \in \text{Im } P$ if and only if

$$(6) \quad f(h_1 g h_2) = f(g)$$

for all $h_1, h_2 \in H$.

For $p_i = Hg_iH$, $g_i \in G$, $i = 1, 2$, and $f \in \mathcal{C}_b(Q)$ viewed as a continuous bounded function on G satisfying (6), let

$$\Delta f(p_1, p_2) = \int_H f(\tilde{p}_1 h \tilde{p}_2) d\mu_H(h),$$

where $\tilde{p}_i \in p_i$, $i = 1, 2$, set $e = H$, $q^* = Hg^{-1}H$, and define μ on Q by

$$\int_Q f(q) d\mu(q) = \int_G f(g) d\mu_G(g)$$

for $q = HgH$, $g \in G$, $f \in \mathcal{C}_c(G)$ satisfying (6). All the axioms are easily verified.

3. CONVOLUTION OF MEASURES

Lemma 3.1. Consider $\mathcal{C}_b(Q)$ and $\mathcal{C}_b(Q \times Q)$ as C^* -algebras. Then $\Delta: \mathcal{C}_b(Q) \rightarrow \mathcal{C}_b(Q \times Q)$ is continuous and $\|\Delta\| = 1$.

Proof. Let $f \in \mathcal{C}_b(Q)$, $f > 0$, be such that $\|f\| = \sup_{p \in Q} |f(p)| = 1$. Then $1 - f \geq 0$ in $\mathcal{C}_b(Q)$ and, as follows from (H_1) (b) and (H_1) (c),

$$\Delta(1 - f)(p, q) = 1 - \Delta f(p, q) \geq 0,$$

hence, $0 \leq \Delta f(p, q) \leq 1$ for all $p, q \in Q$. This means that $\|\Delta f\| \leq 1$, and so $\|\Delta\| \leq 1$. On the other hand, $\Delta(1) = 1$, whence the claim. \square

Definition 3.2. Let $\mu, \mu' \in \mathcal{M}(Q)$ be such that the linear functional $\mu * \mu'$, defined by

$$(7) \quad (\mu * \mu')(f) = \int_{Q^2} \Delta(f)(p, q) d\mu(p) d\mu'(q), \quad f \in \mathcal{C}_c(Q),$$

is a measure. Then the measures μ and μ' are called *convolvable*.

Lemma 3.3. If $\mu, \mu' \in \mathcal{M}_b(Q)$, then the measures μ and μ' , as well as the measures μ' and μ are convolvable. The same is true if $\mu' \in \mathcal{M}_c(Q)$ and $\mu \in \mathcal{M}(Q)$.

Proof. Let $\mu, \mu' \in \mathcal{M}_b(Q)$ and $f \in \mathcal{C}_c(Q)$. Then $|f(q)| \leq \|f\|$ for all $q \in Q$ and, by Lemma 3.1, $|\Delta f(p, q)| \leq \|f\|$ for any $p, q \in Q$. Hence,

$$\begin{aligned} |(\mu * \mu')(f)| &= \left| \int_{Q^2} (\Delta f)(p, q) d\mu(p) d\mu'(q) \right| \\ &\leq \int_{Q^2} |(\Delta f)(p, q)| d|\mu|(p) d|\mu'|(q) \end{aligned}$$

$$\leq \|f\| \int_Q d|\mu|(p) \int_Q d|\mu'|(q),$$

which shows that $\mu * \mu'$ is a Radon measure.

If now $\mu \in \mathcal{M}(Q)$ and $\mu' \in \mathcal{M}_c(Q)$, then let $g \in \mathcal{C}_c(Q)$ be such that $g(q) = 1$ for $q \in \text{supp } \mu'$. Then $\mu' = g\mu'$ and

$$\begin{aligned} |(\mu * \mu')(f)| &= \left| \int_{Q^2} (\Delta f)(p, q) g(q) d\mu(p) d\mu'(q) \right| \\ &\leq \int_{Q^2} |(\Delta f)(p, q) g(q)| d|\mu|(p) d|\mu'|(q) \\ &\leq \|f\| (|\mu| \times |\mu'|)(K), \end{aligned}$$

where $K \subset Q^2$ is the support of the function $(\Delta f)(1 \otimes g)$, which is compact. This ends the proof. \square

Since the Dirac measure ε_p , $p \in Q$, has compact support, it is convolvable with any measure, and any measure is convolvable with it.

Let μ be an element of $\mathcal{M}(Q)$, $\mathcal{M}_b(Q)$ or $\mathcal{M}_c(Q)$. Then a measure $\check{\mu}$ defined by

$$(8) \quad \check{\mu}(f) = \int_Q f(p^*) d\mu(p)$$

for $f \in \mathcal{C}_c(Q)$ belongs to $\mathcal{M}(Q)$, $\mathcal{M}_b(Q)$ or $\mathcal{M}_c(Q)$, respectively. The same is true for a measure μ^* defined by $\mu^* = \overline{\check{\mu}}$, that is,

$$(9) \quad \mu^*(f) = \overline{\int_Q \bar{f}(p^*) d\mu(p)}$$

Proposition 3.4. *The normed linear space $\mathcal{M}_b(Q)$ is an involutive Banach algebra with respect to the convolution and involution defined by (7) and (9), correspondingly. The measure ε_e is an identity.*

Proof. It follows from Lemma 3.3 that $\mu_1 * \mu_2 \in \mathcal{M}_b(Q)$ for $\mu_1, \mu_2 \in \mathcal{M}_b(Q)$. Moreover, $\mathcal{M}_b(Q)$ is a Banach space with respect to the norm $\|\cdot\|$ [13]. Associativity of convolution follows immediately from axiom (H_1) (a). Indeed, for $\mu_1, \mu_2, \mu_3 \in \mathcal{M}_b(Q)$, $f \in \mathcal{C}_c(Q)$,

$$\begin{aligned} ((\mu_1 * \mu_2) * \mu_3)(f) &= \int_{Q^2} (\Delta f)(p, r) d(\mu_1 * \mu_2)(p) d\mu_3(r) \\ &= \int_{Q^3} ((\Delta \times \text{id}) \circ \Delta f)(p, q, r) d\mu_1(p) d\mu_2(q) d\mu_3(r) \\ &= \int_{Q^3} ((\text{id} \times \Delta) \circ \Delta f)(p, q, r) d\mu_1(p) d\mu_2(q) d\mu_3(r) \\ &= \int_{Q^2} \Delta f(p, q) d\mu_1(p) d(\mu_2 * \mu_3)(q) \\ &= (\mu_1 * (\mu_2 * \mu_3))(f). \end{aligned}$$

To see that $\mu \mapsto \mu^*$ is an involution, we use (H_3) ,

$$\begin{aligned} \overline{(\mu_1 * \mu_2)^*(f)} &= \int_Q \bar{f}(p^*) d(\mu_1 * \mu_2)(p) \\ &= \int_{Q^2} (\Delta \bar{f})(p, q) d\mu_1(p) d\mu_2(q) \\ &= \int_{Q^2} (\Delta \bar{f})(q^*, p^*) d\mu_1(p) d\mu_2(q) \end{aligned}$$

$$\begin{aligned}
&= \int_{Q^2} (\overline{\Delta f})(q^*, p^*) d\mu_2(q) d\mu_1(p) \\
&= \overline{(\mu_2^* * \mu_1^*)(f)}.
\end{aligned}$$

It is clear that $(\mu^*)^* = \mu$. For the norm $\|\mu^*\|$, we have

$$\|\mu^*\| = \sup_{\substack{f \in C_c(Q), \\ \|f\| \leq 1}} |\mu^*(f)| = \sup_{\substack{f \in C_c(Q), \\ \|f\| \leq 1}} |\overline{\mu(\check{f})}| = \sup_{\substack{f \in C_c(Q), \\ \|\check{f}\| \leq 1}} |\mu(f)| = \|\mu\|.$$

We immediately get that ε_e is an identity by using axiom (H_2) . \square

Remark 3.5. Definition 2.1 of a locally compact hypergroup generalizes that of a DJS-hypergroup introduced independently by C. Dunkl [7], R. Jewett [9], and R. Spector [8].

4. UNIQUENESS OF THE HAAR MEASURE. THE MODULUS FUNCTION

Lemma 4.1. *Let $f \in C_c(Q)$. If μ is a left Haar measure, then*

$$(10) \quad \int_Q (\Delta f)(p, q) d\mu(q) = \int_Q f(q) d\mu(q)$$

for each fixed $p \in Q$. If ν is a right Haar measure, then

$$(11) \quad \int_Q (\Delta f)(p, q) d\nu(p) = \int_Q f(p) d\nu(p)$$

for each fixed $p \in Q$.

Proof. By using (H_1) (c) and (4) with $g = 1$, we get

$$\int_Q (\Delta f)(p, q) d\mu(q) = \int_Q f(q) (\Delta 1)(p^*, q) d\mu(q) = \int_Q f(q) d\mu(q),$$

which proves (10). Identity (11) is proved similarly using (5). \square

Lemma 4.2. *If μ is a left (resp., right) Haar measure on Q , then $\check{\mu}$ defined by (8) is a right (resp., left) Haar measure.*

Proof. Let μ be a left Haar measure. Using (H_3) and (4) we have

$$\begin{aligned}
\int_Q (\Delta f)(p, q) g(p) d\check{\mu}(p) &= \int_Q (\Delta f)(p^*, q) g(p^*) d\mu(p) \\
&= \int_Q (\Delta \check{f})(q^*, p) \check{g}(p) d\mu(p) \\
&= \int_Q \check{f}(p) (\Delta \check{g})(q, p) d\mu(p) \\
&= \int_Q f(p^*) (\Delta g)(p^*, q^*) d\mu(p) \\
&= \int_Q f(p) (\Delta g)(p, q^*) d\check{\mu}(p).
\end{aligned}$$

For a right Haar measure, the proof is similar. \square

Proposition 4.3. *Let μ be a left Haar measure and ν a measure satisfying (11). Then $\check{\nu}$ is proportional to μ . If ν is a right Haar measure and μ is a measure satisfying (10), then $\check{\mu}$ is proportional to ν .*

Proof. The argument is standard [13]. Let μ be a left Haar measure and ν a measure satisfying (11). Let $f, g \in \mathcal{C}_c(Q)$. Consider the product $\mu(f)\nu(g)$ and use (11), (4) to obtain

$$\begin{aligned} \mu(f)\nu(g) &= \left(\int_Q f(q) d\mu(q) \right) \left(\int_Q g(p) d\nu(p) \right) \\ &= \int_Q d\mu(q) f(q) \int_Q (\Delta g)(p, q) d\nu(p) \\ &= \int_Q d\nu(p) \int_Q f(q) (\Delta g)(p, q) d\mu(q) \\ &= \int_Q d\nu(p) \int_Q (\Delta f)(p^*, q) g(q) d\mu(q) \\ &= \int_Q d\mu(q) g(q) \int_Q (\Delta f)(p^*, q) d\nu(p). \end{aligned}$$

Taking f such that $\mu(f) \neq 0$ and denoting

$$D_f(q) = \frac{1}{\mu(f)} \int_Q (\Delta f)(p^*, q) d\nu(p)$$

we obtain

$$\nu(g) = \mu(gD_f)$$

for all $g \in \mathcal{C}_c(Q)$. This shows that $D_f(q)$ does not depend on f , because taking $f' \in \mathcal{C}_c(Q)$ satisfying $\mu(f') \neq 0$, we would have from the latter identity that

$$\mu(g(D_f - D_{f'})) = 0$$

for all $g \in \mathcal{C}_c(Q)$. Since μ is positive on open sets and D_f is a continuous function of q , we have $D_f = D_{f'}$. Denoting this function

$$D(q) = \frac{1}{\mu(f)} \int_Q (\Delta f)(p^*, q) d\nu(p)$$

we have

$$D(e) = \frac{1}{\mu(f)} \int_Q (\Delta f)(p^*, e) d\nu(p) = \frac{1}{\mu(f)} \int_Q f(p^*) d\nu(p) = \frac{\check{\nu}(f)}{\mu(f)}.$$

This shows that $\check{\nu}$ is proportional to μ .

The other part is proved similarly. \square

Corollary 4.4. *A left (resp., right) Haar measure is unique up to a constant.*

Proof. A right Haar measure ν satisfies (11), hence any left Haar measure is proportional to $\check{\nu}$. \square

Corollary 4.5. *Let μ be a measure satisfying (10) (resp., (11)). Then μ is a left (resp., right) Haar measure. In other words, if μ is such that $\varepsilon_p * \mu = \mu$ (resp., $\mu * \varepsilon_p = \mu$) for all $p \in Q$, then μ is a left (resp., right) Haar measure.*

Proof. If a measure μ satisfies (10), which is the same as $\varepsilon_p * \mu = \mu$ for all $p \in Q$, then the measure $\check{\mu}$ satisfies (11). Indeed, for $q \in Q$,

$$\begin{aligned} \int_Q (\Delta f)(p, q) d\check{\mu}(p) &= \int_Q (\Delta f)(p^*, q) d\mu(p) = \int_Q (\Delta \check{f})(q^*, p) d\mu(p) \\ &= \int_Q \check{f}(p) d\mu(p) = \int_Q f(p^*) d\mu(p) = \int_Q f(p) d\check{\mu}(p). \end{aligned}$$

This means that a left Haar measure is proportional to $\check{\check{\mu}} = \mu$.

The proof of the other part of the corollary is similar. \square

Lemma 4.1 and Proposition 3.4 show that if μ is a left Haar measure and $p \in Q$, then $\mu * \varepsilon_p$ is a left Haar measure. Since a left Haar measure is unique up to a constant,

$$(12) \quad \mu * \varepsilon_{p^*} = \delta(p)\mu$$

for a positive number $\delta(p)$. This number does not depend on the left Haar measure μ .

Definition 4.6. The function $\delta: Q \rightarrow \mathbb{C}$ defined by (12) is called the *modulus function* of the locally compact hypergroup Q .

Proposition 4.7. *The modulus function δ of Q is positive, continuous and satisfies the following properties:*

- (i) $\delta(p)\delta(p^*) = 1$ for every $p \in Q$;
- (ii) $\check{\mu} = \check{\delta}\mu$ for a left Haar measure μ ;
- (iii) $(\Delta(f\delta^z))(p, q) = (\Delta f)(p, q)\delta^z(p)\delta^z(q)$ for every $f \in \mathcal{C}_c(Q)$ and $z \in \mathbb{C}$.

Proof. For $f \in \mathcal{C}_c(Q)$, $f \geq 0$ and $f(p^*) > 0$, we have that $\Delta f \geq 0$ and, since $(\Delta f)(e, p^*) = f(p^*) > 0$, we have that

$$(\mu * \varepsilon_{p^*})(f) = \int_Q (\Delta f)(q, p^*) d\mu(q) > 0,$$

which shows that $\delta(p) > 0$.

Let $f \in \mathcal{C}_c(Q)$ be such that $\mu(f) \neq 0$. Then consider

$$\delta(p)\mu(f) = \int_Q f(q) d(\mu * \varepsilon_{p^*})(q) = \int_Q (\Delta f)(p, q^*) d\mu(p).$$

But $\Delta f \in \mathcal{C}(Q \times Q)$ showing that $\delta \in \mathcal{C}(Q)$.

Let us now prove (ii). For $f, g \in \mathcal{C}_c(Q)$, $\mu(f) \neq 0$, consider

$$\begin{aligned} \mu(f)\check{\mu}(g) &= \int_Q f(p) d\mu(p) \int_Q (\Delta \check{g})(p^*, q) d\mu(q) \\ &= \int_Q d\mu(q) \int_Q f(p) (\Delta g)(q^*, p) d\mu(p) \\ &= \int_Q d\mu(q) \int_Q (\Delta f)(q, p) g(p) d\mu(p) \\ &= \int_Q g(p) d\mu(p) \int_Q (\Delta f)(q, p) d\mu(q) \\ &= \int_Q g(p) (\mu * \varepsilon_p)(f) d\mu(p) \\ &= \int_Q g(p) \delta(p^*) d\mu(p) \int_Q f(q) d\mu(q) \\ &= \mu(f)\mu(g\check{\delta}). \end{aligned}$$

This proves (ii).

Considering (i) we have

$$\mu = \check{\check{\mu}} = (\check{\delta}\mu)^\check{\check{}} = \delta\check{\check{\mu}} = \delta\check{\delta}\mu,$$

which gives (i).

Let us finally prove (iii). First consider the case $z = 1$. We need to show that

$$(\Delta(f\delta))(p, q)\delta^{-1}(p)\delta^{-1}(q) = (\Delta f)(p, q)$$

for $f \in \mathcal{C}_c(Q)$. Take arbitrary $g_1, g_2 \in \mathcal{C}_c(Q)$ and using property (ii) as well as (5), (4) we get the following:

$$\begin{aligned}
& \int_Q \int_Q (\Delta(f\delta))(p, q) g_1(p) g_2(q) \delta^{-1}(p) \delta^{-1}(q) d\mu(p) d\mu(q) \\
&= \int_Q \int_Q (\Delta(f\delta))(p, q) g_1(p) g_2(q) d\check{\mu}(p) d\check{\mu}(q) \\
&= \int_Q g_2(q) d\check{\mu}(q) \int_Q (\Delta(f\delta))(p, q) g_1(p) d\check{\mu}(p) \\
&= \int_Q g_2(q) d\check{\mu}(q) \int_Q f(p) \delta(p) (\Delta g_1)(p, q^*) d\check{\mu}(p) \\
&= \int_Q f(p) \delta(p) d\check{\mu}(p) \int_Q (\Delta g_1)(p, q^*) g_2(q) d\check{\mu}(q) \\
&= \int_Q f(p) \delta(p) d\check{\mu}(p) \int_Q (\Delta g_1)(p, q) \check{g}_2(q) d\mu(q) \\
&= \int_Q f(p) \delta(p) d\check{\mu}(p) \int_Q g_1(q) (\Delta \check{g}_2)(p^*, q) d\mu(q) \\
&= \int_Q g_1(q) d\mu(q) \int_Q f(p) \delta(p) (\Delta \check{g}_2)(p^*, q) d\check{\mu}(p) \\
&= \int_Q g_1(q) d\mu(q) \int_Q f(p) (\Delta \check{g}_2)(p^*, q) d\mu(p) \\
&= \int_Q g_1(q) d\mu(q) \int_Q f(p) (\Delta g_2)(q^*, p) d\mu(p) \\
&= \int_Q g_1(q) d\mu(q) \int_Q (\Delta f)(q, p) g_2(p) d\mu(p) \\
&= \int_Q \int_Q (\Delta f)(p, q) g_1(p) g_2(q) d\mu(p) d\mu(q).
\end{aligned}$$

This finishes the case $z = 1$. By induction, we immediately see that (iii) is true for $z \in \mathbb{N}$. Next, consider a function $\varphi: \mathbb{C} \rightarrow \mathbb{C}$ defined by

$$\varphi(z) = (\Delta(f\delta^z))(p, q) \delta^{-z}(p) \delta^{-z}(q) - (\Delta f)(p, q)$$

for fixed $f \in \mathcal{C}_0(Q)$, $p, q \in Q$. This is an entire function, in particular, it is analytic in $D = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and continuous on \overline{D} , the closure of D . It follows from the above that $\varphi(z) = 0$ for $z \in \mathbb{N}$. Finally, it is easy to see that there exist $M, \alpha > 0$ such that

$$|\varphi(z)| \leq M e^{\alpha \operatorname{Re} z}$$

for all $z \in \overline{D}$. Now it follows from [14, p. 228] that $\varphi = 0$ on D and, hence, on \mathbb{C} . \square

5. INVOLUTIVE BANACH ALGEBRA STRUCTURE ON $L_1(Q, \mu)$

Everywhere in this section, μ denotes a fixed left Haar measure. As in the case of locally compact groups [13], we make the following definition.

Definition 5.1. A *convolution* of functions f and g for $f, g \in \mathcal{C}_c(Q)$ is the function $f * g$ defined by

$$(13) \quad (f \mu) * (g \mu) = (f * g) \mu,$$

where the convolution of the measures in (13) is given by (7). The function f^* defined by

$$(14) \quad (f \mu)^* = f^* \mu$$

is called an *involution* of f , where the involution of a measure is given by (9).

Lemma 5.2. *Let $f, g \in \mathcal{C}_c(Q)$. Then $f * g \in \mathcal{C}_c(Q)$, $f^* \in \mathcal{C}_c(Q)$, and*

$$(15) \quad (f * g)(q) = \int_Q f(p) (\Delta g)(p^*, q) d\mu(p),$$

$$(16) \quad f^*(q) = \bar{f}(q^*) \delta(q^*).$$

Proof. For $\varphi \in \mathcal{C}_c(Q)$, using (4) we have

$$\begin{aligned} ((f\mu) * (g\mu))(\varphi) &= \int_{Q^2} (\Delta\varphi)(p, q) f(p) g(q) d\mu(p) d\mu(q) \\ &= \int_Q f(p) d\mu(p) \int_Q (\Delta\varphi)(p, q) g(q) d\mu(q) \\ &= \int_Q f(p) d\mu(p) \int_Q \varphi(q) (\Delta g)(p^*, q) d\mu(q) \\ &= \int_Q \varphi(q) d\mu(q) \int_Q f(p) (\Delta g)(p^*, q) d\mu(p). \end{aligned}$$

This shows formula (15). It immediately follows from axiom (H_1) (d) that $f * g \in \mathcal{C}_c(Q)$.

Consider (14). Let $\varphi \in \mathcal{C}_c(Q)$. Then by definition (9),

$$\begin{aligned} (f\mu)^*(\varphi) &= \overline{\int_Q \varphi(p^*) f(p) d\mu(p)} \\ &= \int_Q \varphi(p^*) \bar{f}(p) d\mu(p) \\ &= \int_Q \varphi(p) \bar{f}(p^*) d\tilde{\mu}(p) \\ &= \int_Q \varphi(p) \bar{f}(p^*) \delta(p^*) d\mu(p), \end{aligned}$$

where we have used Proposition 4.7 (ii). It is immediate that $f^* \in \mathcal{C}_c(Q)$. \square

Corollary 5.3. *It follows from Proposition 3.4 that $\mathcal{C}_c(Q)$ is an involutive algebra with the multiplication and involution defined by (15) and (16), correspondingly.*

Proposition 5.4. *Let $f, g \in \mathcal{C}_c(Q)$ and, for $\alpha = 1, 2, \infty$, $\|\cdot\|_\alpha$ denote the norm in the corresponding space $L_\alpha(Q, \mu)$. Then*

$$(17) \quad \|f * g\|_\alpha \leq \|f\|_1 \|g\|_\alpha$$

and

$$(18) \quad \|f^*\|_1 = \|f\|_1.$$

Proof. Let us first consider the case $\alpha = 1$ for (17). Using (10) and that $|\Delta g| \leq \Delta|g|$ we have

$$\begin{aligned} \|f * g\|_1 &= \left\| \int_Q f(p) (\Delta g)(p^*, q) d\mu(p) \right\|_1 \\ &= \int_Q d\mu(q) \left| \int_Q f(p) (\Delta g)(p^*, q) d\mu(p) \right| \\ &\leq \int_Q d\mu(q) \int_Q |f(p)| |(\Delta g)(p^*, q)| d\mu(p) \\ &\leq \int_Q d\mu(q) \int_Q |f(p)| (\Delta|g|)(p^*, q) d\mu(p) \\ &= \int_Q d\mu(p) |f(p)| \int_Q (\Delta|g|)(p^*, q) d\mu(q) \end{aligned}$$

$$\begin{aligned}
&= \int_Q d\mu(p) |f(p)| \int_Q |g(q)| d\mu(q) \\
&= \|f\|_1 \|g\|_1.
\end{aligned}$$

Next, consider the case $\alpha = 2$. We will use the Cauchy–Bunyakovskii inequality, the inequality $|\Delta g|^2 \leq \Delta|g|^2$, and (10),

$$\begin{aligned}
\|f * g\|_2^2 &= \left\| \int_Q f(p) (\Delta g)(p^*, q) d\mu(p) \right\|_2^2 \\
&= \int_Q d\mu(q) \int_{Q^2} f(p_1) (\Delta g)(p_1^*, q) \overline{f(p_2) (\Delta g)(p_2^*, q)} d\mu(p_1) d\mu(p_2) \\
&= \left| \int_{Q^2} d\mu(p_1) d\mu(p_2) f(p_1) \overline{f(p_2)} \int_Q (\Delta g)(p_1^*, q) \overline{(\Delta g)(p_2^*, q)} d\mu(q) \right| \\
&\leq \int_{Q^2} d\mu(p_1) d\mu(p_2) |f(p_1)| |f(p_2)| \left| \int_Q (\Delta g)(p_1^*, q) \overline{(\Delta g)(p_2^*, q)} d\mu(q) \right| \\
&\leq \int_{Q^2} d\mu(p_1) d\mu(p_2) |f(p_1)| |f(p_2)| \left(\int_Q |\Delta g|^2(p_1^*, q_1) d\mu(q_1) \right)^{1/2} \\
&\quad \cdot \left(\int_Q |\Delta g|^2(p_2^*, q_2) d\mu(q_2) \right)^{1/2} \\
&\leq \int_{Q^2} d\mu(p_1) d\mu(p_2) |f(p_1)| |f(p_2)| \left(\int_Q (\Delta|g|^2)(p_1^*, q_1) d\mu(q_1) \right)^{1/2} \\
&\quad \cdot \left(\int_Q (\Delta|g|^2)(p_2^*, q_2) d\mu(q_2) \right)^{1/2} \\
&= \int_{Q^2} d\mu(p_1) d\mu(p_2) |f(p_1)| |f(p_2)| \left(\int_Q |g|^2(q_1) d\mu(q_1) \right)^{1/2} \\
&\quad \cdot \left(\int_Q |g|^2(q_2) d\mu(q_2) \right)^{1/2} \\
&= \|f\|_1^2 \|g\|_2^2.
\end{aligned}$$

Finally, let us consider that case where $\alpha = \infty$.

$$\begin{aligned}
\|f * g\|_\infty &= \sup_{q \in Q} \left| \int_Q f(p) (\Delta g)(p^*, q) d\mu(p) \right| \\
&\leq \int_Q |f(p)| \sup_{q \in Q} |(\Delta g)(p^*, q)| d\mu(p) \\
&\leq \int_Q |f(p)| \|\Delta g\|_\infty d\mu(p) \\
&\leq \|f\|_1 \|\Delta g\|_\infty \\
&\leq \|f\|_1 \|g\|_\infty,
\end{aligned}$$

since $\|\Delta\| = 1$.

Consider now identity (18). We have

$$\begin{aligned}
\|f^*\|_1 &= \int_Q |f^*(p)| d\mu(p) \\
&= \int_Q |f(p^*)| \delta(p^*) d\mu(p) \\
&= \int_Q |f(p)| \delta(p) d\check{\mu}(p)
\end{aligned}$$

$$\begin{aligned}
&= \int_Q |f(p)| \delta(p) \delta(p^*) d\mu(p) \\
&= \int_Q |f(p)| d\mu(p) \\
&= \|f\|_1,
\end{aligned}$$

where we used (16) and Proposition 4.7 (ii). \square

Corollary 5.5. *Let $f \in L_1(Q, \mu)$ and $g \in L_\alpha(Q, \mu)$, $\alpha = 1, 2, \infty$. Let $f_n, g_n \in C_c(Q)$ be sequences of functions such that $f_n \rightarrow f$ in $L_1(Q, \mu)$ and $g_n \rightarrow g$ in $L_\alpha(Q, \mu)$ as $n \rightarrow \infty$. Then*

- (i) $\lim_{n \rightarrow \infty} (f_n * g_n)$ exists and is an element of $L_\alpha(Q, \mu)$;
- (ii) $\lim_{n \rightarrow \infty} f_n^*$ exists and is an element of $L_1(Q, \mu)$.

Proof. It follows from (17) that the sequence $f_n * g_n$ is Cauchy and, since $L_\alpha(Q, \mu)$ is complete, the sequence converges. \square

Definition 5.6. Let f, f_n and g, g_n be as in Corollary 5.5. Then the function $\lim_{n \rightarrow \infty} (f_n * g_n)$ is called *convolution* of f and g . The element f^* is an *involution* of f .

Theorem 5.7. *The space $L_1(Q, \mu)$ is an involutive Banach algebra with respect to the convolution and involution defined by Definition 5.6 and formulas (15), (16).*

Proof. The proof immediately follows from Definition 5.1, Proposition 3.4, Definition 5.6, and Corollary 5.5. \square

Theorem 5.8. *The involutive Banach algebra $L_1(Q, \mu)$ has a two-sided approximate identity (e_n) , that is, there exists a sequence of functions $e_n \in L_1(Q, \mu)$, $n \in \mathbb{N}$, such that, for all n , $e_n \geq 0$, $e_n = e_n^*$ almost everywhere, $\|e_n\|_1 = 1$ and $\|e_n * f - f\|_1 \rightarrow 0$, $\|f * e_n - f\|_1 \rightarrow 0$ as $n \rightarrow \infty$ for any $f \in L_1(Q, \mu)$.*

Moreover, for any $f \in C(Q)$,

$$(19) \quad \lim_{n \rightarrow \infty} \int_Q f(r) e_n(r) d\mu(r) = f(e).$$

Proof. Without loss of generality, we can assume that $f \in C_c(Q)$. Next, let V_n , $n \in \mathbb{N}$, be a fundamental system of open relatively compact neighborhoods of e such that $V_n \supseteq V_{n+1}$, $\bigcap_{n \in \mathbb{N}} V_n = \{e\}$. Let $\tilde{e}_n \in C_c(Q)$ be a sequence of functions such that $\tilde{e}_n \geq 0$, $\tilde{e}_n^* = \tilde{e}_n$, $\text{supp } \tilde{e}_n \subset V_n$ and, finally set $e_n = \tilde{e}_n / \mu(\tilde{e}_n)$, having $\mu(e_n) = 1$ for all n .

Hence,

$$\inf_{q \in V_n} f(q) \leq \int_Q f(r) e_n(r) d\mu(r) \leq \sup_{q \in V_n} f(q),$$

which immediately implies (19).

Now, consider $f(p) - (e_n * f)(p)$ and write it as

$$\begin{aligned}
f(p) - (e_n * f)(p) &= f(p) \left(1 - \int_{V_n} e_n(q) d\mu(q) \right) \\
&\quad + \int_{V_n} e_n(q) (f(p) - (\Delta f)(q^*, p)) d\mu(q) \\
&\quad - \int_{Q \setminus V_n} e_n(q) (\Delta f)(q^*, p) d\mu(q).
\end{aligned}$$

The first term is zero by the definition of e_n . The third term is also zero, since $\text{supp } e_n \subset V_n$. Hence,

$$|f(p) - (e_n * f)(p)| \leq \int_{V_n} e_n(q) |f(p) - (\Delta f)(q^*, p)| d\mu(q).$$

The function

$$(p, q) \mapsto e_n(q)(f(p) - (\Delta f)(q^*, p))$$

is continuous, has compact support contained in $F \times V_n \subset F \times V_1$ for all $n \in \mathbb{N}$, where F is some compact subset of Q . Thus consider the function

$$\Phi: (p, q) \mapsto f(p) - (\Delta f)(q^*, p)$$

on $F \times V_1$. We have that $\Phi(p, e) = 0$ for all $p \in F$. Hence, for any $\varepsilon > 0$ there is a neighborhood V_ε of e such that $|\Phi(p, q)| < \varepsilon$ for all $(p, q) \in F \times V_\varepsilon$. Choosing N such that $V_N \subset V_\varepsilon$ we will have $V_n \subset V_N \subset V_\varepsilon$ for all $n > N$ and thus for such n ,

$$\begin{aligned} |f(p) - (e_n * f)(p)| &\leq \int_{V_n} e_n(q) |f(p) - (\Delta f)(q^*, p)| d\mu(q) \\ &\leq \int_{V_n} e_n(q) \sup_{(p, q) \in F \times V_\varepsilon} |\Phi(p, q)| d\mu(q) \\ &\leq \varepsilon \end{aligned}$$

for all $p \in F$. Thus

$$\begin{aligned} \|f - (e_n * f)\|_1 &= \int_Q |f(p) - (e_n * f)(p)| d\mu(p) \\ &= \int_F |f(p) - (e_n * f)(p)| d\mu(p) \\ &\leq \varepsilon \int_F d\mu(p). \end{aligned}$$

This proves that $\|f - (e_n * f)\|_1 \rightarrow 0$ as $n \rightarrow \infty$.

Let us now consider $f - f * e_n$. First of all recalling that $d\check{\mu}$ is a right Haar measure, hence (5) holds, and we have

$$\begin{aligned} (f * e_n)(p) &= \int_Q f(q) (\Delta e_n)(q^*, p) d\mu(q) \\ &= \int_Q \check{f}(q) (\Delta e_n)(q, p) d\check{\mu}(q) \\ &= \int_Q e_n(q) (\Delta \check{f})(q, p^*) d\check{\mu}(q) \\ &= \int_Q e_n(q^*) (\Delta \check{f})(q^*, p^*) d\mu(q) \\ &= \int_Q e_n(q) (\Delta f)(p, q) d\mu(q). \end{aligned}$$

This shows that

$$\begin{aligned} f(p) - (f * e_n)(p) &= f(p) \left(1 - \int_{V_n} e_n(q) d\mu(q)\right) \\ &\quad + \int_{V_n} e_n(q) (f(p) - (\Delta f)(p, q)) d\mu(q) \\ &\quad - \int_{Q \setminus V_n} e_n(q) (\Delta f)(p, q) d\mu(q). \end{aligned}$$

Then we continue the reasoning as above. \square

Remark 5.9. Everywhere in the sequel, we will assume that the approximate identity (e_n) satisfies the following: $\text{supp } e_n \subset V_n$ where $\bigcap_{n \in \mathbb{N}} V_n = \{e\}$.

6. A LEFT HILBERT ALGEBRA STRUCTURE ON $\mathcal{C}_c(Q)$

Recall [15] that a linear subspace \mathfrak{A} of a Hilbert space H is called a *left Hilbert algebra* if \mathfrak{A} is an associative algebra with involution \sharp and the following holds:

- (i) the map $\sharp: \mathfrak{A} \rightarrow \mathfrak{A}$ is a preclosed operator on H ;
- (ii) $(fg, h)_H = (g, f\sharp h)_H$ for $f, g, h \in \mathfrak{A}$;
- (iii) for every $f \in \mathfrak{A}$, the operator $L_f: g \mapsto fg$, $g \in \mathfrak{A}$, can be extended to a continuous operator on H ;
- (iv) $\mathfrak{A} \cdot \mathfrak{A}$ is dense in H .

Proposition 6.1. *Let Q be a locally compact hypergroup. Consider the algebra $\mathfrak{A} = \mathcal{C}_c(Q)$ with multiplication $*$ defined by (15) and involution $*$ defined by (16), with the scalar product induced from the Hilbert space $H = L_2(Q, \mu)$. Then $(\mathfrak{A}, *, *)$ is a left Hilbert algebra.*

Proof. It follows from Corollary 5.3 that $\mathcal{C}_c(Q)$ is an involutive algebra.

To see that the map $*$: $\mathcal{C}_c(Q) \rightarrow \mathcal{C}_c(Q)$ is preclosed in $L_2(Q, \mu)$, let $f_n \in \mathcal{C}_c(Q)$, $f_n \rightarrow 0$ and $f_n^* \rightarrow f$ in $L_2(Q, \mu)$ for $f \in L_2(Q, \mu)$. Since

$$\begin{aligned} \|f_n^* - f\|_2^2 &= \int_Q |f_n^*(q) - f(q)|^2 d\mu(q) \\ &= \int_Q |\bar{f}_n(q^*)\delta(q^*) - f(q)|^2 d\mu(q) = \int_Q |\bar{f}_n(q)\delta(q) - f(q^*)|^2 d\mu(q^*) \\ &= \int_Q |\bar{f}_n(q)\delta(q) - f(q^*)|^2 \delta(q^*) d\mu(q) = \int_Q |f_n(q) - \bar{f}(q^*)\delta(q^*)|^2 \delta(q) d\mu(q). \end{aligned}$$

This show that $f_n \rightarrow f^* = 0$ almost everywhere, hence $f = 0$ in $L_2(Q, \mu)$ and $*$ is preclosed in $L_2(Q, \mu)$.

Consider now the left-hand side of the identity in (ii). We have

$$(f * g, h)_H = \int_Q \bar{h}(q) (f * g)(q) d\mu(q) = \int_Q \bar{h}(q) \left(\int_Q f(r) \Delta g(r^*, q) d\mu(r) \right) d\mu(q).$$

The right-hand side of (ii) will be

$$\begin{aligned} (g, f^* * h)_H &= \int_Q \overline{(f^* * h)(q)} g(q) d\mu(q) \\ &= \int_Q \overline{\left(\int_Q f^*(r) \Delta h(r^*, q) d\mu(r) \right)} g(q) d\mu(q) \\ &= \int_Q \left(\int_Q \delta(r^*) f(r^*) \Delta \bar{h}(r^*, q) d\mu(r) \right) g(q) d\mu(q) \\ &= \int_Q \left(\int_Q f(r) \Delta \bar{h}(r, q) d\mu(r) \right) g(q) d\mu(q) \\ &= \int_Q f(r) \left(\int_Q \Delta \bar{h}(r, q) g(q) d\mu(q) \right) d\mu(r) \\ &= \int_Q f(r) \left(\int_Q \Delta g(r^*, q) \bar{h}(q) d\mu(q) \right) d\mu(r), \end{aligned}$$

where we have used that the measure μ is left-invariant. By comparing the above two expressions, we see that (ii) holds.

Using inequality (17) with $\alpha = 2$, we see that L_f can be extended to a continuous operator with $\|L_f\| \leq \|f\|_1$, hence (iii) is true.

Consider (iv). For $f \in \mathfrak{A}$ and an approximate identity (e_n) , using (17) we have

$$\|f - e_n * f\|_\infty \leq \|f\|_\infty + \|e_n * f\|_\infty \leq \|f\|_\infty + \|e_n\|_1 \|f\|_\infty = 2\|f\|_\infty.$$

Since, for any $g \in \mathfrak{A}$,

$$\|g\|_2^2 = \int_Q |g(q)|^2 d\mu(q) \leq \|g\|_\infty \|g\|_1,$$

we have that

$$\|f - e_n * f\|_2 \leq \|f - e_n * f\|_\infty^{1/2} \|f - e_n * f\|_1^{1/2} \leq (2\|f\|_\infty)^{1/2} \|f - e_n * f\|_1^{1/2},$$

showing that $e_n * f \rightarrow f$ in $L_2(Q, \mu)$. This proves (iv), since $e_n \in \mathfrak{A}$ for all n . \square

Proposition 6.2. *Let S denote the closure of the antilinear operator $*$ and $S = JD^{1/2}$ be its polar decomposition, where D is positive and J is an antilinear isometry. Then, for any $f \in \mathcal{C}_c(Q)$,*

$$(20) \quad (D^{1/2}f)(p) = \delta^{1/2}(p)f(p), \quad (Jf)(p) = \delta^{1/2}(p^*)\bar{f}(p^*).$$

Proof. For $f, g \in \mathcal{C}_c(Q)$, we have

$$\begin{aligned} (Sf, g)_H &= \int_Q f^*(q)\bar{g}(q) d\mu(q) = \int_Q \delta(q^*)\bar{f}(q^*)\bar{g}(q) d\mu(q) \\ &= \int_Q \delta(q)\bar{f}(q)\bar{g}(q^*) d\mu(q^*) = \int_Q \bar{f}(q)\bar{g}(q^*) d\mu(q). \end{aligned}$$

Hence, $S^*(g)(p) = \bar{g}(p^*)$ and $(Df)(p) = (S^*Sf)(p) = (\overline{Sf})(p^*) = \delta(p)f(p)$. This proves the first identity in (20).

To prove the second identity in (20), consider $g = \delta^{1/2}f = D^{1/2}f$. We have

$$(Jg)(p) = (JD^{1/2}f)(p) = f^*(p) = \delta(p^*)\bar{f}(p^*) = \delta(p^*)\delta^{-1/2}(p^*)\bar{g}(p^*) = \delta^{1/2}(p^*)\bar{g}(p^*).$$

\square

Corollary 6.3. *For $f \in \mathcal{C}_c(Q)$, set*

$$(21) \quad f^\dagger(p) = \bar{f}(p^*).$$

Then $(f^\dagger)^\dagger = f$ and

$$(22) \quad (f, g * h)_H = (f * h^\dagger, g)_H,$$

Proof. It follows from Proposition 6.2 that the involution adjoint to $*$ is \dagger , hence we get the result [15]. \square

7. REPRESENTATIONS OF A LOCALLY COMPACT HYPERGROUP

Lemma 7.1. *For each $p \in Q$, let $L_p: \mathcal{C}_c(Q) \rightarrow \mathcal{C}_c(Q)$ and $R_p: \mathcal{C}_c(Q) \rightarrow \mathcal{C}_c(Q)$ be defined by*

$$(23) \quad (L_p f)(q) = (\Delta f)(p^*, q),$$

$$(24) \quad (R_p f)(q) = (\Delta f)(q, p) \delta^{\frac{1}{2}}(p).$$

Then $\|L_p f\|_\alpha \leq \|f\|_\alpha$, $\alpha \in \{1, 2\}$, and $\|R_p f\|_2 \leq \|f\|_2$, where $\|\cdot\|_\alpha$ is the norm in $L_\alpha(Q, \mu)$, μ is a left Haar measure.

Proof. Consider the case $\alpha = 1$. Let $p \in Q$ be fixed. Since $|\Delta f| \leq \Delta|f|$, using (10) we have

$$\|L_p f\|_1 = \int_Q |L_p f|(r) d\mu(r) = \int_Q |\Delta f(p^*, r)| d\mu(r)$$

$$\leq \int_Q \Delta |f|(p^*, r) d\mu(r) = \int_Q |f|(r) d\mu(r) = \|f\|_1.$$

Now consider the case $\alpha = 2$. Since $|\Delta f|^2 \leq \Delta |f|^2$, we have the following estimate for a fixed $p \in Q$:

$$\begin{aligned} \|L_p f\|_2^2 &= \int_Q |(L_p f)(q)|^2 d\mu(q) \\ &= \int_Q |\Delta f|^2(p^*, q) d\mu(q) \\ &\leq \int_Q (\Delta |f|^2)(p^*, q) d\mu(q) \\ &= \int_Q |f|^2(q) d\mu(q) \\ &= \|f\|_2^2. \end{aligned}$$

Now, consider $\|R_p f\|_2^2$ and use properties (ii), (iii) and (i) in Proposition 4.7,

$$\begin{aligned} \|R_p f\|_2^2 &= \int_Q |(\Delta f(q, p) \delta^{\frac{1}{2}}(p))|^2 d\mu(q) \\ &= \delta(p) \int_Q |\Delta f|^2(q, p) d\mu(q) \\ &\leq \delta(p) \int_Q (\Delta |f|^2)(q, p) d\mu(q) \\ &= \delta(p) \int_Q (\Delta |f|^2)(q, p) \delta(q) d\tilde{\mu}(q) \\ &= \delta(p) \delta(p^*) \int_Q \Delta(|f|^2 \delta)(q, p) d\tilde{\mu}(q) \\ &= \int_Q |f|^2(q) \delta(q) d\tilde{\mu}(q) \\ &= \|f\|_2^2. \end{aligned}$$

□

Corollary 7.2. For $p \in Q$, the operator $L_p: L_1(Q, \mu) \rightarrow L_1(Q, \mu)$ and the operators $L_p, R_p: L_2(Q, \mu) \rightarrow L_2(Q, \mu)$ have norms 1. Hence they can be extended by continuity to the corresponding Banach and Hilbert spaces.

Definition 7.3. The extension of L_p (resp. R_p) by continuity to the Hilbert space $L_2(Q, \mu)$ is called a *generalized left* (resp. *right*) *translation operator*. We will still denote it by L_p (resp. R_p).

Definition 7.4. Let H be a Hilbert space, $\mathcal{L}(H)$ the algebra of all bounded operators on H . A weakly continuous mapping $\pi: Q \rightarrow \mathcal{L}(H)$ is called a *representation* of the hypergroup Q if it satisfies the following properties:

- (i) $\pi(e) = I$, the identity operator on H ;
- (ii) $\pi(p^*) = \pi(p)^*$;
- (iii) for every $\xi, \eta \in H$,

$$(25) \quad \Delta(\pi(\cdot)\xi, \eta)_H(p, q) = (\pi(p)\pi(q)\xi, \eta)_H.$$

A representation π is called *bounded* if the function $p \mapsto \|\pi(p)\|$ is bounded on Q .

Proposition 7.5. *Let μ be a left Haar measure on Q , and set $H = L_2(Q, \mu)$. Then the mappings*

$$\pi_L: p \mapsto L_p, \quad \pi_R: p \mapsto R_p$$

are bounded representations of Q . Moreover, they separate points, that is, if $p \neq q$, then $\pi_L(p) \neq \pi_L(q)$ and $\pi_R(p) \neq \pi_R(q)$.

Proof. For $f, g \in \mathcal{C}_c(Q)$, we have

$$\begin{aligned} (L_p f, g)_H &= \int_Q \Delta f(p^*, q) \bar{g}(q) d\mu(q), \\ (R_p f, g)_H &= \delta^{\frac{1}{2}}(p) \int_Q \Delta f(q, p) \bar{g}(q) d\mu(q), \end{aligned}$$

whence weak continuity of π_L and π_R follows, since $\Delta f(1 \otimes \bar{g})$ and $\Delta f(\bar{g} \otimes 1)$ are functions continuous on $Q \times Q$ and have compact support, and $\delta^{\frac{1}{2}} \in \mathcal{C}(Q)$.

Property (i) for π_L (resp. π_R) follows immediately from (H_2) and the definitions of π_L (resp. π_R).

Property (ii) for π_L is equivalent to axiom (H_4) . Indeed, let $f, g \in \mathcal{C}_c(Q)$. Then

$$\begin{aligned} (L_p f, g)_H &= \int_Q (L_p f)(q) \bar{g}(q) d\mu(q) \\ &= \int_Q \Delta f(p^*, q) \bar{g}(q) d\mu(q). \end{aligned}$$

On the other hand,

$$\begin{aligned} (f, L_{p^*} g)_H &= \int_Q f(q) \overline{L_{p^*} g}(q) d\mu(q) \\ &= \int_Q f(q) \Delta \bar{g}(p, q) d\mu(q). \end{aligned}$$

Hence, the identity $L_p^* = L_{p^*}$ is indeed equivalent to (H_4) . To see that (ii) holds for R_p , we will use positivity of δ , Proposition 4.7 (i)–(iii), and that $\check{\mu}$ is a right invariant measure thus satisfying (5). Take $f, g \in \mathcal{C}_c(Q)$ and consider

$$\begin{aligned} (R_p f, g)_H &= \int_Q \Delta f(q, p) \delta^{\frac{1}{2}}(p) \bar{g}(q) d\mu(q) \\ &= \int_Q \Delta f(q, p) \bar{g}(q) \delta^{\frac{1}{2}}(p) \delta(q) d\check{\mu}(q) \\ &= \int_Q f(q) \Delta(\bar{g} \delta)(q, p^*) \delta^{\frac{1}{2}}(p) d\check{\mu}(q) \\ &= \int_Q f(q) \overline{\Delta g}(q, p^*) \delta(q) \delta(p^*) \delta^{\frac{1}{2}}(p) \delta^{-1}(q) d\mu(q) \\ &= \int_Q f(q) \overline{\Delta g}(q, p^*) \delta^{-\frac{1}{2}}(p) d\mu(q) \\ &= \int_Q f(q) \overline{R_{p^*} g}(q) d\mu(q) \\ &= (f, R_{p^*} g)_H. \end{aligned}$$

Let us now consider (iii) in Definition 7.4. Let $f, g \in \mathcal{C}_c(Q)$ and $p, q \in Q$. Consider π_L . On the one hand,

$$(L_p(L_q f), g)_H = \int_Q (L_p(L_q f))(r) \bar{g}(r) d\mu(r) = \int_Q \Delta(L_q f)(p^*, r) \bar{g}(r) d\mu(r)$$

$$= \int_Q ((\text{id} \times \Delta) \circ \Delta f)(q^*, p^*, r) \bar{g}(r) d\mu(r).$$

On the other hand,

$$\begin{aligned} \Delta(L_\bullet f, g)_H(p, q) &= \Delta\left(\int_Q (L_\bullet f)(r) \bar{g}(r) d\mu(r)\right)(p, q) \\ &= \int_Q \Delta(L_\bullet f)(p, q, r) \bar{g}(r) d\mu(r) \\ &= \int_Q ((\Delta \times \text{id}) \circ (\sim \times \text{id}) \circ \Delta f)(p, q, r) \bar{g}(r) d\mu(r) \\ &= \int_Q ((\Delta \times \text{id}) \circ \Delta f)(q^*, p^*, r) \bar{g}(r) d\mu(r). \end{aligned}$$

This proves (iii) for π_L .

Consider now π_R . Let $f, g \in \mathcal{C}_c(Q)$. The right-hand side of (25) becomes

$$\begin{aligned} (R_p(R_q f, g))_H &= \int_Q R_p(R_q f)(r) \bar{g}(r) d\mu(r) \\ &= \int_Q \Delta(R_q f)(r, p) \delta^{\frac{1}{2}}(p) \bar{g}(r) d\mu(r) \\ &= \int_Q ((\Delta \times \text{id}) \circ \Delta f)(r, p, q) \delta^{\frac{1}{2}}(q) \delta^{\frac{1}{2}}(p) \bar{g}(r) d\mu(r). \end{aligned}$$

For the left-hand side of (25), using Proposition 4.7 (iii) with $z = 1/2$, we have

$$\begin{aligned} \Delta(R_\bullet f, g)_H(p, q) &= \Delta\left(\int_Q (R_\bullet f)(r) \bar{g}(r) d\mu(r)\right)(p, q) \\ &= \int_Q \Delta(R_\bullet f)(r, p, q) \bar{g}(r) d\mu(r) \\ &= \int_Q \Delta(\Delta f(r, \cdot) \delta^{\frac{1}{2}})(p, q) \bar{g}(r) d\mu(r) \\ &= \int_Q \Delta(\Delta f(r, \cdot))(p, q) \delta^{\frac{1}{2}}(p) \delta^{\frac{1}{2}}(q) \bar{g}(r) d\mu(r). \end{aligned}$$

This shows that π_R satisfies (25).

Since $\|L_p\| \leq 1$ and $\|R_p\| \leq 1$ for $p \in Q$ by Corollary 7.2, the representations π_L and π_R are bounded.

To show that π_L separates points, let $p, q \in Q$, $p \neq q$. Then there is a function $f \in \mathcal{C}_c(Q)$ such that $f(p^*) \neq f(q^*)$. Thus $L_p(f)(e) = \Delta(f)(p^*, e) = f(p^*)$ and, similarly, $L_q(f)(e) = f(q^*)$. This means that $L_p(f) \neq L_q(f)$ in $L_2(Q, \mu)$, hence $\pi_L(p) \neq \pi_L(q)$.

For π_R , the proof is similar. □

Lemma 7.6. *Let $p \in Q$ and $f, g \in \mathcal{C}_c(Q)$. Then*

$$(26) \quad L_p(f * g) = L_p(f) * g,$$

$$(27) \quad (L_p(f))^* * g = f^* * (L_{p^*}(g)).$$

Proof. Let us first prove identity (26). For $p, q \in Q$, using the definition of L_p , coassociativity of Δ , right-invariance of $\check{\mu}$, property (H_3) we have

$$L_p(f * g)(q) = L_p\left(\int_Q f(r) (\Delta g)(r^*, \cdot) d\mu(r)\right)(q)$$

$$\begin{aligned}
&= \Delta \left(\int_Q f(r) (\Delta g)(r^*, \cdot) d\mu(r) \right) (p^*, q) \\
&= \int_Q f(r) ((\text{id} \times \Delta) \circ \Delta g)(r^*, p^*, q) d\mu(r) \\
&= \int_Q f(r) ((\Delta \times \text{id}) \circ \Delta g)(r^*, p^*, q) d\mu(r) \\
&= \int_Q \check{f}(r^*) ((\Delta \times \text{id}) \circ \Delta g)(r^*, p^*, q) d\mu(r) \\
&= \int_Q \check{f}(r) ((\Delta \times \text{id}) \circ \Delta g)(r, p^*, q) d\check{\mu}(r) \\
&= \int_Q \Delta \check{f}(r, p) \cdot \Delta g(r, q) d\check{\mu}(r) = \int_Q \Delta \check{f}(r^*, p) \cdot \Delta g(r^*, q) d\mu(r) \\
&= \int_Q \Delta f(p^*, r) \cdot \Delta g(r^*, q) d\mu(r).
\end{aligned}$$

On the other hand,

$$(L_p(f) * g)(q) = \int_Q L_p(f)(r) \cdot \Delta g(r^*, q) d\mu(r) = \int_Q \Delta f(p^*, r) \cdot \Delta g(r^*, q) d\mu(r),$$

which ends the proof of (26).

Consider now (27). The left-hand side equals

$$\begin{aligned}
((L_p(f))^* * g)(q) &= \int_Q (L_p(f))^*(r) \Delta g(r^*, q) d\mu(r) \\
&= \int_Q \overline{L_p(f)}(r^*) \delta(r^*) \Delta g(r^*, q) d\mu(r) = \int_Q \Delta \bar{f}(p^*, r^*) \Delta g(r^*, q) \delta(r^*) d\mu(r) \\
&= \int_Q \Delta \bar{f}(p^*, r) \Delta g(r, q) \delta(r) d\mu(r^*) = \int_Q \Delta \bar{f}(p^*, r) \Delta g(r, q) d\mu(r) \\
&= \int_Q \bar{f}(r) ((\Delta \times \text{id}) \circ \Delta g)(p, r, q) d\mu(r),
\end{aligned}$$

where we have used Proposition 4.7 and left invariance of μ .

Write out the right-hand side of (27) as

$$\begin{aligned}
(f^* * L_{p^*}(g))(q) &= \int_Q f^*(r) \Delta(L_{p^*}(g))(r^*, q) d\mu(r) \\
&= \int_Q \bar{f}(r^*) \delta(r^*) ((\text{id} \times \Delta) \circ \Delta g)(p, r^*, q) d\mu(r) \\
&= \int_Q \bar{f}(r) ((\text{id} \times \Delta) \circ \Delta g)(p, r, q) \delta(r) d\mu(r^*) \\
&= \int_Q \bar{f}(r) ((\text{id} \times \Delta) \circ \Delta g)(p, r, q) d\mu(r).
\end{aligned}$$

Since Δ is coassociative, we see that both sides are equal. \square

Theorem 7.7. *Let $\pi: Q \rightarrow \mathcal{L}(H)$ be a bounded nondegenerate representation of the hypergroup Q on a Hilbert space H . Then $\hat{\pi}: L_1(Q, \mu) \rightarrow \mathcal{L}(H)$ given by*

$$(28) \quad \hat{\pi}(f) = \int_Q f(q) \pi(q) d\mu(q), \quad f \in L_1(Q, \mu),$$

defines a nondegenerate representation $\hat{\pi}$ of the Banach algebra $L_1(Q, \mu)$ on H , where the integral is understood in the sense of Bochner [16].

Conversely, let $\hat{\pi}: L_1(Q, \mu) \rightarrow \mathcal{L}(H)$ be a nondegenerate representation of the Banach algebra $L_1(Q, \mu)$, (e_n) an approximate identity, $p \in Q$, and L_p is considered as a mapping $L_1(Q, \mu) \rightarrow L_1(Q, \mu)$. Then the limit

$$(29) \quad \pi(p) = \lim_{n \rightarrow \infty} \hat{\pi}(L_p(e_n))$$

exists in the strong operator topology, and the mapping $\pi: Q \rightarrow \mathcal{L}(H)$ defined by (29) is a nondegenerate bounded representation of the hypergroup Q . Moreover,

$$(30) \quad \pi(p) \hat{\pi}(f) \xi = \hat{\pi}(L_p(f)) \xi,$$

where $f \in L_1(Q, \mu)$, $\xi \in H$.

Proof. Let π be a representation of Q on H and show that $\hat{\pi}$ is a representation of $L_1(Q, \mu)$. We start by showing that $\hat{\pi}(f) \in \mathcal{L}(H)$ for $f \in L_1(Q, \mu)$. Indeed,

$$\begin{aligned} \|\hat{\pi}(f)\| &= \left\| \int_Q \pi(q) f(q) d\mu(q) \right\| \leq \int_Q \|\pi(q) f(q)\| d\mu(q) \\ &= \int_Q \|\pi(q)\| \cdot |f(q)| d\mu(q) \leq \sup_{q \in Q} \|\pi(q)\| \cdot \int_Q |f(q)| d\mu(q) \\ &= \sup_{q \in Q} \|\pi(q)\| \cdot \|f\|_1 < \infty, \end{aligned}$$

since π is a bounded representation of Q .

Next, consider $\hat{\pi}(f^*)$. We have

$$\begin{aligned} \hat{\pi}(f^*) &= \int_Q \pi(q) f^*(q) d\mu(q) = \int_Q \pi(q) \bar{f}(q^*) \delta(q^*) d\mu(q) \\ &= \int_Q \pi(q^*) \bar{f}(q) \delta(q) d\mu(q^*) = \int_Q \pi(q)^* \bar{f}(q) \delta(q) \delta(q^*) d\mu(q) \\ &= \left(\int_Q \pi(q) f(q) d\mu(q) \right)^* = \hat{\pi}(f)^*, \end{aligned}$$

where we have used the definition (16) of f^* and Proposition 4.7.

Consider now

$$\hat{\pi}(f_1 * f_2) = \int_Q \pi(q) (f_1 * f_2)(q) d\mu(q) = \int_Q \pi(q) \left(\int_Q f_1(r) \Delta(f_2)(r^*, q) d\mu(r) \right) d\mu(q).$$

For $\xi, \eta \in H$, we have

$$\begin{aligned} (\hat{\pi}(f_1 * f_2) \xi, \eta)_H &= \int_Q (\pi(q) \xi, \eta)_H \int_Q f_1(r) \Delta(f_2)(r^*, q) d\mu(r) d\mu(q) \\ &= \int_Q f_1(r) \int_Q (\pi(q) \xi, \eta)_H \Delta(f_2)(r^*, q) d\mu(q) d\mu(r) \\ &= \int_Q f_1(r) \int_Q \Delta((\pi(\cdot) \xi, \eta)_H)(r, q) f_2(q) d\mu(q) d\mu(r) \\ &= \int_Q f_1(r) \int_Q (\pi(r) \pi(q) \xi, \eta)_H f_2(q) d\mu(q) d\mu(r) \\ &= \int_Q f_1(r) (\pi(r) \hat{\pi}(f_2) \xi, \eta)_H d\mu(r) = (\hat{\pi}(f_1) \hat{\pi}(f_2) \xi, \eta)_H, \end{aligned}$$

where we have used (iii) of Definition 7.4.

Consider now the converse and prove that the limit in (29) exists in the strong operator topology. Since $\hat{\pi}$ is a representation of a Banach algebra, it is continuous and $\|\hat{\pi}(f)\| \leq$

$\|f\|_1$ for $f \in L_1(Q, \mu)$. Since it is nondegenerate, the set $\hat{\pi}(A)H$ is dense in H . Hence, consider a vector of the form $\hat{\pi}(f)\xi$ for some $f \in A$, $\xi \in H$, $\|\xi\|_H = 1$. For $n \in \mathbb{N}$, using (26) and Theorem 5.8 we have

$$\begin{aligned} \|\hat{\pi}(L_p(e_n))\hat{\pi}(f)\xi - \hat{\pi}(L_p(f))\xi\|_H &= \|\hat{\pi}(L_p(e_n) * f - L_p(f))\xi\|_H \\ &\leq \|\hat{\pi}(L_p(e_n) * f - L_p(f))\| \cdot \|\xi\|_H \leq \|L_p(e_n) * f - L_p(f)\|_1 \\ &= \|L_p(e_n * f) - L_p(f)\|_1 = \|L_p(e_n * f - f)\|_1 \\ &\leq \|e_n * f - f\|_1 \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

proving (30). To see that $\pi(p)$ is bounded on H , let $\xi \in H$, $\|\xi\|_H = 1$. Then

$$\begin{aligned} \|\pi(p)\xi\|_H &= \left\| \lim_{n \rightarrow \infty} \hat{\pi}(L_p(e_n))\xi \right\|_H = \lim_{n \rightarrow \infty} \|\hat{\pi}(L_p(e_n))\xi\|_H \\ &\leq \lim_{n \rightarrow \infty} \|L_p(e_n)\|_1 \|\xi\|_H = \lim_{n \rightarrow \infty} \|e_n\|_1 = 1. \end{aligned}$$

This, in particular, shows that the map $p \mapsto \|\pi(p)\|$ is bounded.

Let us now prove that (29) defines a representation of Q on H .

Since $L_e(e_n) = e_n$ and (e_n) is an approximate identity, $\hat{\pi}(L_e(e_n)) \rightarrow I$ strongly on H .

To prove that $\pi(p)^* = \pi(p^*)$, take $\eta = \hat{\pi}(f)\xi \in H$, $f \in L_1(Q, \mu)$, and using Theorem 5.8 and (27) consider

$$\begin{aligned} \hat{\pi}(L_p(e_n))^* \eta &= \hat{\pi}(L_p(e_n)^*) \hat{\pi}(f)\xi = \hat{\pi}(L_p(e_n)^* * f)\xi \\ &= \hat{\pi}(e_n^* * L_{p^*}(f))\xi = \hat{\pi}(e_n * L_{p^*}(f))\xi \xrightarrow{n \rightarrow \infty} \hat{\pi}(L_{p^*}(f))\xi, \end{aligned}$$

where the limit is taken in H . This shows that $\lim_{n \rightarrow \infty} \hat{\pi}(L_p(e_n))^*$ exists on each vector of a dense subset of H and is bounded. Hence, for $\eta = \hat{\pi}(f)\xi$,

$$\pi(p)^* \eta = \left(\lim_{n \rightarrow \infty} \hat{\pi}(L_p(e_n)) \right)^* \eta = \lim_{n \rightarrow \infty} (\hat{\pi}(L_p(e_n))^* \eta) = \hat{\pi}(L_{p^*}(f))\xi.$$

On the other hand,

$$\begin{aligned} \hat{\pi}(L_{p^*}(e_n))\eta &= \hat{\pi}(L_{p^*}(e_n))\hat{\pi}(f)\xi = \hat{\pi}(L_{p^*}(e_n) * f)\xi \\ &= \hat{\pi}(L_{p^*}(e_n * f))\xi \xrightarrow{n \rightarrow \infty} \hat{\pi}(L_{p^*}(f))\xi. \end{aligned}$$

And this shows that $\pi(p)^* = \pi(p^*)$.

To prove that (iii) in Definition 7.4 is satisfied, for $p, q, r \in Q$, consider

$$\begin{aligned} (\Delta(L_\bullet(f))(p, q))(r) &= (\Delta(L_\bullet(f)))(p, q, r) = ((\Delta \times \text{id}) \circ (\sim \times \text{id}) \circ \Delta f)(p, q, r) \\ &= ((\Delta \times \text{id}) \circ \Delta f)(q^*, p^*, r) = (L_p(L_q(f)))(r). \end{aligned}$$

Hence, for $\eta_1 = \hat{\pi}(f)\xi \in H$, $f \in L_1(Q, \mu)$, $\eta_2 \in H$, we have

$$\begin{aligned} (\Delta(\pi(\cdot)\eta_1, \eta_2)_H)(p, q) &= (\Delta(\hat{\pi}(L_\bullet(f))\xi, \eta_2)_H)(p, q) \\ &= (\hat{\pi}(\Delta(L_\bullet(f))(p, q))\xi, \eta_2)_H = (\hat{\pi}(L_p(L_q(f)))\xi, \eta_2)_H, \end{aligned}$$

as follows from the preceding identity. On the other hand,

$$\begin{aligned} (\pi(p)\pi(q)\eta_1, \eta_2)_H &= (\pi(p)\pi(q)\hat{\pi}(f)\xi, \eta_2)_H \\ &= (\pi(p)\hat{\pi}(L_q(f))\xi, \eta_2)_H = (\hat{\pi}(L_p(L_q(f)))\xi, \eta_2)_H, \end{aligned}$$

which ends the proof of (iii), Definition 7.4.

It is clear that nondegenerate representations of $L_1(Q, \mu)$ correspond to nondegenerate representations of Q and vice versa. \square

8. AN APPROXIMATION THEOREM

Definition 8.1. A continuous bounded function k on Q is called *positive definite* if for any $n \in \mathbb{N}$, $q_i \in Q$, $i = 1, \dots, n$, the matrix

$$(\Delta k(q_i^*, q_j))_{1 \leq i, j \leq n}$$

is positive semi-definite [17].

Definition 8.2. For two positive definite functions k_1, k_2 , we say that k_1 *majorizes* k_2 , written by $k_1 \succ k_2$, if $k_1 - k_2$ is positive definite. A positive definite function k is called *elementary* if any positive definite function majorized by k is of the form λk , $\lambda \in [0, 1]$.

Lemma 8.3. Let A denote the involutive Banach algebra $L_1(Q, \mu)$. A continuous bounded function k on Q is positive definite if and only if the functional $\chi_k: A \rightarrow \mathbb{C}$ defined by

$$(31) \quad \chi_k(f) = \int_Q k(q) f(q) d\mu(q)$$

is positive on A , that is,

$$(32) \quad \chi_k(f^* * f) = \int_Q k(q) (f^* * f) d\mu(q) \geq 0$$

for any $f \in A$.

Proof. First of all, let us show that

$$(33) \quad \chi_k(g^* * f) = \int_{Q^2} \Delta k(r^*, q) \bar{g}(r) f(q) d\mu(r) d\mu(q).$$

Indeed,

$$\begin{aligned} \int_Q k(q) (g^* * f)(q) d\mu(q) &= \int_Q k(q) d\mu(q) \int_Q g^*(r) \Delta f(r^*, q) d\mu(r) \\ &= \int_Q k(q) d\mu(q) \int_Q \bar{g}(r^*) \delta(r^*) \Delta f(r^*, q) d\mu(r) \\ &= \int_Q \bar{g}(r^*) \delta(r^*) d\mu(r) \int_Q k(q) \Delta f(r^*, q) d\mu(q) \\ &= \int_Q \bar{g}(r^*) \delta(r^*) d\mu(r) \int_Q \Delta k(r, q) f(q) d\mu(q) \\ &= \int_{Q^2} \Delta k(r, q) \bar{g}(r^*) f(q) \delta(r^*) d\mu(r) d\mu(q) \\ &= \int_{Q^2} \Delta k(r^*, q) \bar{g}(r) f(q) d\mu(r) d\mu(q). \end{aligned}$$

Thus, for $g = f$, we have

$$\chi_k(f^* * f) = \int_{Q^2} \Delta k(r^*, q) \bar{f}(r) f(q) d\mu(r) d\mu(q).$$

So, let the functional χ_k on A be positive, i.e., (32) hold. Let $q_i \in Q$, $i = 1, \dots, n$, be given. Let $\varepsilon > 0$ and, for each i , choose a neighborhood U_i about q_i such that

$$|\Delta k(r_i^*, r_j) - \Delta k(q_i^*, q_j)| < \varepsilon \quad \forall (r_i, r_j) \in U_i \times U_j.$$

and $U_i \cap U_j = \emptyset$ if $i \neq j$. Let $c_i \in \mathbb{C}$, $i = 1, \dots, n$, be arbitrary complex numbers. Denoting by f_i the characteristic function of the set U_i , set

$$f = \sum_{i=1}^n \frac{c_i}{\mu(U_i)} f_i.$$

Then

$$\begin{aligned} \chi_k(f^* * f) &= \int_{Q^2} \Delta k(r^*, q) \sum_{i=1}^n \frac{\bar{c}_i}{\mu(U_i)} f_i(q) \sum_{j=1}^n \frac{c_j}{\mu(U_j)} f_j(r) d\mu(r) d\mu(q) \\ &= \sum_{i,j=1}^n \frac{\bar{c}_i c_j}{\mu(U_i) \mu(U_j)} \int_{U_i \times U_j} \Delta k(r^*, q) d\mu(r) d\mu(q) \\ &= \sum_{i,j=1}^n \frac{\bar{c}_i c_j}{\mu(U_i) \mu(U_j)} (\Delta k(q_i^*, q_j) + \varepsilon_{ij}) \mu(U_i) \mu(U_j) \\ &= \sum_{i,j=1}^n \bar{c}_i c_j (\Delta k(q_i^*, q_j) + \varepsilon_{ij}), \end{aligned}$$

where $|\varepsilon_{ij}| < \varepsilon$. Hence, positivity of the functional χ_k implies that the matrix

$$(\Delta k(q_i^*, q_j) + \varepsilon_{ij})_{1 \leq i, j \leq n}$$

is positive definite. Since ε is arbitrary, the matrix $(\Delta k(q_i^*, q_j))_{1 \leq i, j \leq n}$ itself is positive definite.

Let now k be positive definite and prove (32). Take $f \in \mathcal{C}_c(Q)$ with compact support K . For arbitrary $q_i \in K$, $i = 1, \dots, n$, we have

$$\sum_{i,j=1}^n \Delta k(q_i^*, q_j) \bar{f}(q_i) f(q_j) \geq 0.$$

Hence,

$$\begin{aligned} &\int_{K^n} \left(\sum_{i,j=1}^n \Delta k(q_i^*, q_j) \bar{f}(q_i) f(q_j) \right) d\mu(q_1) \dots d\mu(q_n) \\ &= n \mu(K)^{n-1} \int_K \Delta k(q^*, q) \bar{f}(q) f(q) d\mu(q) \\ &\quad + n(n-1) \mu(K)^{n-2} \int_{Q^2} \Delta k(r^*, q) \bar{f}(r) f(q) d\mu(r) d\mu(q) \geq 0. \end{aligned}$$

This means that

$$\frac{\mu(K)}{n-1} \int_Q \Delta k(q^*, q) \bar{f}(q) f(q) d\mu(q) + \int_{Q^2} \Delta k(r^*, q) \bar{f}(r) f(q) d\mu(r) d\mu(q) \geq 0.$$

Making $n \rightarrow \infty$, we see that

$$\int_{Q^2} \Delta k(r^*, q) \bar{f}(r) f(q) d\mu(r) d\mu(q) \geq 0,$$

which proves (32) for f with compact support and, hence, for any $f \in L_1(Q, \mu)$. \square

Corollary 8.4. *For continuous bounded positive definite k_1, k_2 , we have $k_1 \succ k_2$ if and only if $\chi_{k_1} \succ \chi_{k_2}$.*

Theorem 8.5. *A continuous function k on Q is bounded and positive definite if and only if there is a Hilbert space H_k , a bounded representation π_k of Q on H_k , and a vector $\xi_k \in H_k$ such that*

$$(34) \quad k(q) = (\pi_k(q)\xi_k, \xi_k)_{H_k}, \quad q \in Q.$$

The representation π_k is irreducible if and only if k is elementary.

Proof. Let π be a bounded representation of Q on H , and k be defined by (34).

For $q_i \in Q$, $c_i \in \mathbb{C}$, $i = 1, \dots, n$, and using property (25) we have

$$\begin{aligned} \sum_{i,j=1}^n (\Delta k(q_i^*, q_j)) \bar{c}_i c_j &= \sum_{i,j=1}^n (\Delta(\pi_k(\cdot)\xi_k, \xi_k)_{H_k}(p_i^*, p_j)) \bar{c}_i c_j \\ &= \sum_{i,j=1}^n (\pi_k(q_i^*)\pi_k(q_j)\xi_k, \xi_k)_{H_k} \bar{c}_i c_j = \sum_{i,j=1}^n (\pi_k(q_j)\xi_k, \pi_k(q_i)\xi_k)_{H_k} \bar{c}_i c_j \\ &= \left(\sum_{j=1}^n c_j \pi_k(q_j)\xi_k, \sum_{i=1}^n c_i \pi_k(q_i)\xi_k \right)_{H_k} = \left\| \sum_{i=1}^n c_i \pi_k(q_i)\xi_k \right\|_{H_k}^2 \geq 0, \end{aligned}$$

which means that the matrix $(\Delta k(q_i^*, q_j))_{1 \leq i, j \leq n}$ is positive definite, hence such is the function k .

Conversely, let k be a bounded positive definite function. This means that the functional χ_k defined in (31) is positive and bounded on the involutive Banach algebra $A = L_1(Q, \mu)$. Using the GNS-construction, see [18], we obtain a Hilbert space H_k , a representation $\hat{\pi}_k$ of A and a cyclic vector ξ_k such that

$$\chi_k(f) = (\hat{\pi}_k(f)\xi_k, \xi_k)_{H_k}, \quad f \in A.$$

By setting $f = L_p(e_n)$ and making $n \rightarrow \infty$, see (30), we get

$$\begin{aligned} (\hat{\pi}_k(p)\xi_k, \xi_k)_{H_k} &= \lim_{n \rightarrow \infty} \chi_k(L_p(e_n)) = \lim_{n \rightarrow \infty} \chi_k(L_p(e_n^*)) = \lim_{n \rightarrow \infty} \int_Q k(r) L_p(e_n^*)(r) d\mu(r) \\ &= \lim_{n \rightarrow \infty} \int_Q k(r) \Delta e_n(r^*, p) d\mu(r) = \lim_{n \rightarrow \infty} (k * e_n)(p) = k(p). \end{aligned}$$

It is clear that $\pi: Q \rightarrow \mathcal{L}(H)$ is weakly continuous if and only if k is continuous, and π is bounded if and only if k is bounded.

The representation π_k is irreducible if and only if χ_k is pure [18] and this is the case, by Corollary 8.4, if and only if k is elementary. □

Lemma 8.6. *Let k be a continuous bounded positive definite function on Q and χ_k the associated positive functional defined by (31) on $L_1(Q, \mu)$. Then*

- (i) $k(e) = \|\chi_k\|$;
- (ii) $|k(p)| \leq k(e)$ for any $p \in Q$.
- (iii) $k(p^*) = \overline{k(p)}$;
- (iv) $|k(p) - \Delta k(p, q)|^2 \leq 2k(e) \operatorname{Re}(k(e) - k(q))$ for all $p, q \in Q$.

Proof. For the positive functional χ_k on the Banach involutive algebra $A = L_1(Q, \mu)$, consider $\hat{\pi}_k$, H_k , ξ_k , which are the corresponding representation, the Hilbert space, and the vector obtained via the GNS-construction. Then [18], since A has an approximate identity by Theorem 5.8,

$$\|\chi_k\| = \|\xi_k\|^2 = k(e),$$

which proves (i).

Consider (ii). Using Lemma 7.1 we have that $\|L_p(e_n)\|_1 \leq \|e_n\|_1 = 1$. Hence,

$$\begin{aligned} |k(p)| &= \left| \lim_{n \rightarrow \infty} (\hat{\pi}_k(L_p(e_n))\xi_k, \xi_k)_{H_k} \right| = \left| \lim_{n \rightarrow \infty} \chi_k(L_p(e_n)) \right| \\ &\leq \sup_{\|f\|_1 \leq 1} |\chi_k(f)| = \|\chi_k\| = k(e), \end{aligned}$$

proving (ii).

Next, consider the corresponding representation of Q on H_k . We have

$$k(p^*) = (\pi(p^*)\xi_k, \xi_k)_{H_k} = (\xi_k, \pi(p)\xi_k)_{H_k} = \overline{(\pi(p)\xi_k, \xi_k)_{H_k}} = \overline{k(p)},$$

which gives (iii).

To prove (iv), consider

$$\begin{aligned} |k(p) - \Delta k(p, q)|^2 &= |(\pi(p)\xi_k, \xi_k)_{H_k} - (\pi(p)\pi(q)\xi_k, \xi_k)_{H_k}|^2 \\ &= |(\pi(p)(I - \pi(q))\xi_k, \xi_k)_{H_k}|^2 = |((I - \pi(q))\xi_k, \pi(p^*)\xi_k)_{H_k}|^2 \\ &\leq \|(I - \pi(q))\xi_k\|_{H_k}^2 \|\pi(p^*)\xi_k\|_{H_k}^2. \end{aligned}$$

It follows from (ii) that $-k(e) \leq k(p) \leq k(e)$ for all $p \in Q$. Hence, $-k(e) \leq \Delta k(p, q) \leq k(e)$, that is, $|\Delta k(p, q)| \leq k(e)$ for all $p, q \in Q$. Thus, we have the following estimate for the second factor:

$$\|\pi(p^*)\xi_k\|_{H_k}^2 = (\pi(p)\pi(p^*)\xi_k, \xi_k)_{H_k} = \Delta k(p, p^*) \leq k(e).$$

For the first factor, we have

$$\begin{aligned} \|(I - \pi(q))\xi_k\|_{H_k}^2 &= ((I - \pi(q))\xi_k, (I - \pi(q))\xi_k)_{H_k} \\ &= \|\xi_k\|_{H_k}^2 + \|\pi(q)\xi_k\|_{H_k}^2 - 2\operatorname{Re}(\pi(q)\xi_k, \xi_k)_{H_k} \\ &\leq 2k(e) - 2\operatorname{Re}(\pi(q)\xi_k, \xi_k)_{H_k} = 2\operatorname{Re}(k(e) - k(q)), \end{aligned}$$

which ends the proof. \square

Corollary 8.7. *For $f \in \mathcal{C}_c(Q)$, let f^\dagger be defined by (21). Then $f * f^\dagger$ is a positive definite function.*

Proof. Indeed, for any $g \in \mathcal{C}_c(Q)$, using property (ii) in the definition of a left Hilbert algebra and (22) we have

$$\begin{aligned} \chi_{f * f^\dagger}(g^* * g) &= \int_Q (f * f^\dagger)(\overline{g^* * g})(p) d\mu(p) \\ &= (f * f^\dagger, \overline{g^* * g})_H = (\overline{g} * f, \overline{g} * f)_H = \|\overline{g} * f\|_H^2 \geq 0. \end{aligned}$$

Hence, $\chi_{f * f^\dagger}$ is a positive functional, thus $f * f^\dagger$ is positive definite. \square

Lemma 8.8. *For any $f, g \in \mathcal{C}_c(Q)$, $f * g^\dagger$ is a weighted sum of four positive definite functions, namely,*

$$(35) \quad 4f * g^\dagger = (f+g)*(f+g)^\dagger - (f-g)*(f-g)^\dagger + i(f+ig)*(f+ig)^\dagger - i(f-ig)*(f-ig)^\dagger.$$

Proof. The proof is immediate. \square

Lemma 8.9. *Let $f \in L_1(Q, \mu)$ be fixed. Then the map $L_\bullet(f): Q \rightarrow L_1(Q, \mu)$ defined by*

$$(36) \quad Q \ni p \mapsto L_p(f) \in L_1(Q, \mu)$$

is continuous.

Proof. Let $p_0 \in Q$ and prove that the map (36) is continuous in the point p_0 . Fix $\varepsilon > 0$, and let $f_0 \in \mathcal{C}_c(Q)$ be such that $\|f - f_0\|_1 < \varepsilon$. Take a neighborhood U of p_0 that has compact closure, $E = \overline{U}$. Let $g \in \mathcal{C}_c(Q)$ be such that $g(p^*) = 1$ for $p \in E$. Since the function f_0 has compact support, $(g \otimes 1)\Delta f_0$ also has compact support, which we denote by K , $K \subset Q \times Q$. Let F be the image of K under the projection of $Q \times Q$ onto the second factor. Thus F is also compact. Finally, since Δf_0 is continuous on the compact set K , choose a neighborhood U_1 of the point p_0 such that $|\Delta f_0(p^*, q) - \Delta f_0(p_0^*, q)| < \varepsilon/\mu(F)$ for all $p \in U_1$ and all $q \in F$. Hence, for $p \in U \cap U_1$, we have

$$\begin{aligned} \|L_p(f_0) - L_{p_0}(f_0)\|_1 &= \int_Q |\Delta f_0(p^*, q) - \Delta f_0(p_0^*, q)| d\mu(q) \\ &= \int_Q |(\Delta f_0(p^*, q) - \Delta f_0(p_0^*, q))g(p^*)| d\mu(q) \\ &= \int_F |\Delta f_0(p^*, q) - \Delta f_0(p_0^*, q)| d\mu(q) < \frac{\varepsilon}{\mu(F)}\mu(F) = \varepsilon. \end{aligned}$$

Thus, for $p \in U \cap U_1$, we have

$$\begin{aligned} \|L_p(f) - L_{p_0}(f)\|_1 &\leq \|L_p(f) - L_p(f_0)\|_1 + \|L_p(f_0) - L_{p_0}(f_0)\|_1 + \|L_{p_0}(f_0) - L_{p_0}(f)\|_1 \\ &= \|L_p(f - f_0)\|_1 + \|L_p(f_0) - L_{p_0}(f_0)\|_1 + \|L_{p_0}(f_0 - f)\|_1 \\ &\leq \|f - f_0\|_1 + \|L_p(f_0) - L_{p_0}(f_0)\|_1 + \|f_0 - f\|_1 < 3\varepsilon, \end{aligned}$$

which ends the proof. \square

Corollary 8.10. *Denote by $\sigma(L_1(Q, \mu), L_\infty(Q, \mu))$ the weak topology on $L_1(Q, \mu)$ induced by the dual space $L_\infty(Q, \mu)$. Then the map given by (36) is weakly continuous.*

Proof. Indeed, for any $g \in L_\infty(Q, \mu)$, $f \in L_1(Q, \mu)$, and $p, p_0 \in Q$, we have

$$\int_Q g(q)(L_p(f) - L_{p_0}(f))(q) d\mu(q) \leq \|g\|_\infty \|L_p(f) - L_{p_0}(f)\|_1.$$

The continuity follows now from Lemma 8.9. \square

Lemma 8.11. *Let $f \in \mathcal{C}_0(Q)$ and (e_n) be an approximate identity. Then*

$$f - f * e_n \rightarrow 0, \quad n \rightarrow \infty,$$

uniformly on compact subsets of Q .

Proof. Let K be a compact subset of Q . Denote by E_n the support of e_n and, for $p \in K$, consider

$$\begin{aligned} |f(p) - (f * e_n)(p)| &= \left| f(p) - \int_Q f(q)\Delta e_n(q^*, p) d\mu(q) \right| \\ &= \left| f(p) - \int_Q f(q)\Delta \check{e}_n(p^*, q) d\mu(q) \right| \\ &= \left| \int_Q f(p)e_n(q) d\mu(q) - \int_Q \Delta f(p, q)e_n(q) d\mu(q) \right| \\ &= \left| \int_{E_n} (\Delta f(p, e) - \Delta f(p, q))e_n(q) d\mu(q) \right| \\ &\leq \sup_{q \in E_n} |\Delta f(p, e) - \Delta f(p, q)| \int_{E_n} e_n(q) d\mu(q) = \sup_{q \in E_n} |\Delta f(p, e) - \Delta f(p, q)|. \end{aligned}$$

Since K and E_n are compact for all n , $E_n \subset E_{n_1}$ if $n > n_1$, $\bigcap_{n \in \mathbb{N}} E_n = \{e\}$, and Δf is continuous, for a given $\varepsilon > 0$ there is $n_1 \in \mathbb{N}$ such that

$$|f(p) - (f * e_n)(p)| < \varepsilon$$

for all $n > n_1$ and $p \in K$. \square

Proposition 8.12. *Let P_1 be the set of continuous bounded positive definite functions k on Q such that $k(e) = 1$. Then the weak $\sigma(L_1(Q, \mu), L_\infty(Q, \mu))$ -topology and the topology of uniform convergence on compact sets coincide on P_1 .*

Proof. It is clear that if k_0 is a limit point of a set X , $X \subset P_1$, in the topology of uniform convergence on compact sets, then it is a limit point with respect to the weak topology. Let us prove the converse.

Assume that k_0 is a limit point of X with respect to the weak topology. Fix a compact set $K \subset Q$, take an arbitrary $k \in P_1$ and let (e_n) be an approximate identity. Then we have

$$(37) \quad |k_0(p) - k(p)| \leq |(k_0 - k_0 * e_n)(p)| + |((k_0 - k) * e_n)(p)| + |(k * e_n - k)(p)|.$$

Fix now $\varepsilon > 0$. Since k_0 is continuous and $k_0(e) = 1$ because $k_0 \in P_1$, choose a neighborhood V of e such that $|1 - k_0(p)| < \varepsilon$ for all $p \in V$. By Lemma 8.11, the first term in (37) approaches zero uniformly on K , hence there is n such that

$$|(k_0 - k_0 * e_n)(p)| < \varepsilon$$

for all $p \in K$. Clearly, we can assume that $V_n = \text{supp } e_n \subset V$.

Use Lemma 8.11 to find a $\sigma(L_1(Q, \mu), L_\infty(Q, \mu))$ -weak neighborhood \mathcal{U}_1 of the point k_0 such that

$$|((k - k_0) * e_n)(p)| < \varepsilon$$

for all $k \in \mathcal{U}_1$ and any $p \in K$.

Finally, let \mathcal{U}_2 be a $\sigma(L_1(Q, \mu), L_\infty(Q, \mu))$ -weak neighborhood of the point k_0 such that

$$\left| \int_Q (k_0 - k)(q) e_n(q) d\mu(q) \right| < \varepsilon$$

for any $k \in \mathcal{U}_2$ and the e_n found above.

Let $k \in \mathcal{U}_1 \cap \mathcal{U}_2 \cap X$ and consider the third term in (37). Using (4), (iv) in Lemma 8.6, and that $\check{e}_n = e_n$ and $\|e_n\|_1 = 1$ we have

$$\begin{aligned} |(k * e_n)(p) - k(p)| &= \left| \int_Q k(q) \Delta e_n(q^*, p) d\mu(q) - k(p) \right| \\ &= \left| \int_Q k(q) \Delta e_n(p^*, q) d\mu(q) - k(p) \right| \\ &= \left| \int_V \Delta k(p, q) e_n(q) d\mu(q) - \int_V k(p) e_n(q) d\mu(q) \right| \\ &\leq \int_V |\Delta k(p, q) - k(p)| e_n(q) d\mu(q) \\ &\leq \sqrt{2} \int_V (\text{Re}(1 - k(q)))^{1/2} e_n(q)^{1/2} e_n(q)^{1/2} d\mu(q) \\ &\leq \sqrt{2} \left(\int_V \text{Re}(1 - k(q)) e_n(q) d\mu(q) \right)^{1/2} \left(\int_V e_n(q) d\mu(q) \right)^{1/2} \\ &= \sqrt{2} \left(\int_V \text{Re}(1 - k_0(q)) e_n(q) d\mu(q) + \text{Re} \int_V (k_0 - k)(q) e_n(q) d\mu(q) \right)^{1/2} \\ &\leq \sqrt{2} \left(\int_V |1 - k_0(q)| e_n(q) d\mu(q) + \left| \int_V (k_0 - k)(q) e_n(q) d\mu(q) \right| \right)^{1/2} \end{aligned}$$

$$\leq \sqrt{2}(\varepsilon + \varepsilon)^{1/2} = 2\varepsilon^{1/2},$$

since $|1 - k_0(p)| \leq \varepsilon$ for $p \in V$ and because $k \in \mathcal{U}_2$.

This shows that for any $\varepsilon > 0$ there is $k \in X$ such that

$$|k_0(p) - k(p)| < 2(\varepsilon + \varepsilon^{1/2})$$

for any $p \in K$, and this ends the proof. \square

Theorem 8.13. *Every continuous function can be uniformly approximated on a compact set with linear combinations of elementary positive definite functions.*

Proof. Fix a compact subset K of Q and let $f \in \mathcal{C}(Q)$. We can assume that $f \in \mathcal{C}_0(Q)$. Using Lemma 8.11 we can approximate f with $f * e_n$ uniformly on K , where (e_n) is an approximate identity. Using identity (35) we can represent $f * e_n$ as a weighted sum of four positive definite functions f_i , $i = 1, \dots, 4$, and due to Lemma 8.6 (ii) we can assume that $f_i(e) = 1$, that is $f_i \in P_1$. The set \tilde{P}_1 of continuous positive definite functions k such that $k(e) \leq 1$ is a convex $\sigma(L_\infty(Q, \mu), L_1(Q, \mu))$ -weakly compact subset of the locally convex space $L_\infty(Q, \mu)$. Since the extreme points of \tilde{P}_1 are precisely the elementary positive definite functions, applying the Krein-Milman theorem, we approximate, with respect to the $\sigma(L_\infty(Q, \mu), L_1(Q, \mu))$ weak topology, each function f_i with a finite weighted sum of elementary positive definite functions. Application of Proposition 8.12 ends the proof. \square

9. A PLANCHEREL THEOREM AND INVERSION FORMULAS FOR A LOCALLY COMPACT HYPERGROUP

In this section, \mathfrak{A} denotes the left Hilbert algebra considered in Section 6.

Theorem 9.1. *Let \mathcal{L} (resp. \mathcal{R}) denote the von Neumann algebra generated by the operators L_p (resp. R_p), $p \in Q$, on $H = L_2(Q, \mu)$. Then $\mathcal{L}' = \mathcal{R}$ and $\mathcal{R}' = \mathcal{L}$.*

Proof. It follows from Theorem 7.7 that the von Neumann algebra \mathcal{L} coincides with the von Neumann algebra $\pi_L(\mathfrak{A})''$ generated by the left regular representation of \mathfrak{A} on H , so that $\mathcal{L}' = \pi_L(\mathfrak{A})'$. However, it follows from [15] that $\pi_L(\mathfrak{A})' = (J\pi_L(\mathfrak{A})J)''$. Whence, to prove the theorem, it is sufficient to show that

$$JL_p J = R_p$$

on \mathfrak{A} . Indeed, for $f \in \mathfrak{A}$, using (20) and Proposition 4.7 (iii) with $z = 1/2$ we have

$$\begin{aligned} (JL_p J f)(q) &= \delta^{\frac{1}{2}}(q^*) \overline{(L_p J f)}(q^*) = \delta^{\frac{1}{2}}(q^*) \Delta(\overline{J f})(p^*, q^*) \\ &= \delta^{\frac{1}{2}}(q^*) \Delta(\delta^{\frac{1}{2}} \check{f})(p^*, q^*) = \delta^{\frac{1}{2}}(q^*) \Delta(\delta^{\frac{1}{2}} f)(q, p) \\ &= \delta^{\frac{1}{2}}(q^*) \delta^{\frac{1}{2}}(q) \delta^{\frac{1}{2}}(p) \Delta f(q, p) = \delta^{\frac{1}{2}}(p) \Delta f(q, p) = (R_p f)(q). \end{aligned}$$

\square

Let $H = L_2(Q, \mu)$ and denote by $L_f = \hat{\pi}_L(f)$, $f \in \mathfrak{A}$, the operator defined by (28) for the left regular representation π_L of Q defined in Proposition 7.5. Let \mathcal{L} be the von Neumann algebra generated by L_f , $f \in \mathfrak{A}$, on H and φ the weight on \mathcal{L} corresponding to the scalar product in H , i.e., it is defined by

$$(38) \quad \varphi(L_g^* L_f) = (f, g)_H, \quad f, g \in \mathfrak{A}.$$

Let H_φ be the Hilbert space obtained from \mathcal{L} and φ via the GNS-construction. The central decomposition theorem for von Neumann algebras applied to \mathcal{L} gives

$$(39) \quad H_\varphi = \int_Z^\oplus H_\varphi(z) d\rho(z), \quad \mathcal{L} = \int_Z^\oplus \mathcal{L}(z) d\rho(z), \quad \varphi = \int_Z^\oplus \varphi_z d\rho(z),$$

where Z is spectrum of the center of \mathcal{L} .

Definition 9.2. The measure ρ on Z will be called a *Plancherel* measure. The Fourier transform \hat{f} of $f \in \mathfrak{A}$ is defined on Z by

$$(40) \quad \hat{f}(z) = L_f(z) = \int_Q f(q) L_q(z) d\mu(q), \quad z \in Z.$$

Theorem 9.3. Let the Fourier transform be given by (40) and ρ a Plancherel measure. Then

$$(41) \quad (f, g)_H = \int_Z \varphi_z(\hat{g}^*(z)\hat{f}(z)) d\rho(z)$$

$$(42) \quad f(q) = \int_Z \varphi_z(L_q(z)^*\hat{f}(z)) d\rho(z),$$

and the Fourier transform $\hat{\cdot}$ can be extended to a unitary operator $L_2(Q, \mu) \rightarrow H_\varphi$.

Proof. Identity (41) follows immediately from (38) that defines the weight φ and the decomposition (39). Indeed, we have

$$(f, g)_H = \varphi(L_g^*L_f) = \int_Z \varphi_z(L_g^*(z)L_f(z)) d\rho(z) = \int_Z \varphi_z(\hat{g}^*(z)\hat{f}(z)) d\rho(z),$$

showing that the Fourier transform is an isometry. Since H_φ is the closure of the image of the Fourier transform, we see that it is a unitary operator.

Let $f \in \mathfrak{A}$, with the Fourier image \hat{f} and let

$$f_1(q) = \int_Z \varphi_z(L_q^*(z)\hat{f}(z)) d\rho(z).$$

Then, for any $g \in \mathfrak{A}$, we have

$$\begin{aligned} (f_1, g)_H &= \int_Q \bar{g}(r)f_1(r) d\mu(r) = \int_Q \left(\bar{g}(r) \int_Z \varphi_z(L_r^*(z)\hat{f}(z)) d\rho(z) \right) d\mu(r) \\ &= \int_Z \varphi_z \left(\int_Q \bar{g}(r)L_r^*(z) d\mu(r) \hat{f}(z) \right) d\rho(z) = \int_Z \varphi_z(L_g^*(z)L_f(z)) d\rho(z) \\ &= \varphi(L_g^*L_f) = (f, g)_H. \end{aligned}$$

This shows that $f - f_1$ is orthogonal to any $g \in \mathfrak{A}$, hence it is zero. \square

Remark 9.4. The idea to use the central decomposition of the von Neumann algebra generated by the left regular representation to obtain a Plancherel formula for generalized translation operators is due to L. I. Vainerman and G. L. Litvinov [19].

10. HARMONIC ANALYSIS ON A COCOMMUTATIVE HYPERGROUP

The results in this section were obtained in [11] in the terminology of hypercomplex systems. Here we give them for the sake of completeness. The proofs are easily adopted from [11].

Definition 10.1. A hypergroup Q is called *Hermitian* if $q^* = q$ for all $q \in Q$; it is called *cocommutative* if

$$\Delta f(p, q) = \Delta f(q, p),$$

for all $f \in \mathcal{C}_b(Q)$, $p, q \in Q$.

Remark 10.2. It directly follows from (H_3) that a Hermitian hypergroup is cocommutative.

Definition 10.3. A function $\chi \in \mathcal{C}_b(Q)$ is called a *character* of the hypergroup Q if $(\Delta\chi)(p, q) = \chi(p)\chi(q)$ for all $p, q \in Q$. A character χ is called *Hermitian* if $\chi(p^*) = \overline{\chi(p)}$, $p \in Q$.

For the rest of this subsection, the hypergroup Q is assumed to be cocommutative.

Let X_h be the space of bounded Hermitian characters. For each character $\chi \in X_h$, its kernel is a maximal ideal. In what follows, we endow X_h with the topology of the space of maximal ideals of the involutive Banach algebra $L_1(Q, \mu)$.

Theorem 10.4. *Every continuous positive definite function k on Q can be uniquely represented as an integral,*

$$(43) \quad k(p) = \int_{X_h} \chi(p) d\rho(\chi),$$

with respect to some nonnegative finite Borel measure ρ on the space X_h . Conversely, every function of the form (43) is continuous and positive definite.

Definition 10.5. For a function $f \in L_1(Q, \mu)$, the function $\hat{f}: X_h \rightarrow \mathbb{C}$ defined by

$$(44) \quad \hat{f}(\chi) = \int_Q f(p) \bar{\chi}(p) d\mu(p),$$

is called the Fourier transform of f .

Theorem 10.6. *The Fourier transform given by (44) defines a unitary operator of the space $L_2(Q, \mu)$ onto the space $L_2(\hat{Q}, \rho)$, where the measure ρ on \hat{X}_h , called a Plancherel measure, is uniquely defined, $\hat{Q} = \text{supp } \rho$, and the following inversion formula holds:*

$$(45) \quad f(p) = \int_{\hat{Q}} \hat{f}(\chi) \chi(p) d\rho(\chi).$$

Remark 10.7. This theorem is now a simple corollary of Theorem 9.3. Let us remark that the Fourier transform maps $L_1(Q, \mu)$ into $\mathcal{C}_0(\hat{Q})$ and the inverse Fourier transform, given by formula (45), maps $L_1(\hat{Q}, \rho)$ into $\mathcal{C}_0(Q)$.

Theorems 10.4 and 10.6 give the following.

Corollary 10.8. *A function $k \in L_1(Q, \mu) \cap \mathcal{C}_b(Q)$ is positive definite if and only if $\hat{k}(\chi) \geq 0$ for all $h \in \hat{Q}$.*

The following lemma directly follows from Theorem 10.4.

Lemma 10.9. *Let $\chi_1 \cdot \chi_2$ be a positive definite function on Q for all $\chi_1, \chi_2 \in \hat{Q}$. Then there exists a nonnegative finite regular Borel measure ρ_{χ_1, χ_2} on X_h such that*

$$(46) \quad \chi_1(p)\chi_2(p) = \int_{X_h} \chi(p) d\rho_{\chi_1, \chi_2}(\chi).$$

Theorem 10.10. *Let Q be a cocommutative hypergroup satisfying the following properties:*

- (1) *the character ϵ defined in (H_2) belongs to \hat{Q} ;*
- (2) *the product of two characters $\chi_1, \chi_2 \in \hat{Q}$ is a positive definite function, and the support of the measure μ_{χ_1, χ_2} defined by (46) is contained in \hat{Q} ;*
- (3) *the comultiplication $\hat{\Delta}: \mathcal{C}_b(\hat{Q}) \rightarrow \mathcal{C}_b(\hat{Q} \times \hat{Q})$ defined by*

$$(47) \quad \hat{\Delta}(F)(\chi_1, \chi_2) = \int_{\hat{Q}} F(\chi) d\mu_{\chi_1, \chi_2}(\chi), \quad F \in \mathcal{C}_b(\hat{Q}),$$

satisfies axiom (H_1) (d).

Then \hat{Q} is also a locally compact commutative hypergroup, a so-called dual hypergroup, that satisfies the conditions of this theorem, and the hypergroup $\hat{\hat{Q}}$ dual to \hat{Q} coincides with Q . The hypergroup dual to a compact hypergroup is a discrete hypergroup, the hypergroup dual to a discrete hypergroup is a compact hypergroup.

Theorem 10.11. *Let conditions of Theorem 10.10 be satisfied. Then*

$$A(Q) = L_2(Q, \mu) * L_2(Q, \mu)$$

is a Banach algebra (an analogue of the Fourier algebra) with respect to the pointwise multiplication and the norm $\|f\|_{A(Q)} = \|\hat{f}\|_{L_1(\hat{Q}, \hat{\mu})}$.

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