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Convergence of equilibria for numerical approximations of a suspension model

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ABSTRACT

In this paper we study the numerical approximations of a non-Newtonian model for concentrated suspensions.

First, we prove that the approximative models possess a unique fixed point and study their convergence to a stationary point of the original equation.

Second, we implement an implicit Euler scheme, proving the convergence of these approximations as well.

Finally, numerical simulations are provided.

1. Introduction

Non-Newtonian (or complex) fluids often appear in nature and industry. Good examples of such fluids are toothpaste, ketchup, magma, blood, mucus or emulsions such as mayonnaise among many others. A special type of complex fluids are concentrated suspensions, which can be found, for example, in medicine (blood) or in building industry (cement). The dynamical behaviour of suspensions is still far from being well understood as developing a faithful mathematical model of such processes is not an easy task.

We are interested in an equation modelling suspensions which was proposed in [1]. In the last years, several authors have studied for this equation the existence and uniqueness of solutions [2,3], the asymptotic behaviour [4,5] and numerical approximations [6–8].

In our previous paper [8] we studied a sequence of approximative problems for this model, in which finite-difference schemes were used to deal with the partial derivative with respect to the spatial variable. The problem was split in three steps: a partial differential equation with a large diffusion, an infinite system of ordinary differential equations and finally a finite system of ordinary differential equations. For initial data satisfying suitable assumptions it was proved that the iterate limit of the solutions of the approximative problems in the space $C([0,T], L^2(\mathbb{R}))$ is equal to the solution of the original equation.

In this paper we extend the results from [8] in two ways.

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First, we study the convergence of the fixed points of the approximative problems. It is well-known [2] that for certain values of the parameters of the equation there exists a unique fixed point of the problem with support not included in the interval $[-1, 1]$. This equilibrium is asymptotically stable [4] and the numerical simulations in [7] suggest that every solution with initial data with support not included in $[-1, 1]$ converges to this fixed point as time goes to $+\infty$. We prove that each of the approximative problems possesses a unique fixed point and also that the iterate limit of the equilibria of the approximative problems in the space $L^2(\mathbb{R})$ is equal to the equilibrium of the original equation with support not included in the interval $[-1, 1]$.

Second, we complete the sequence of approximations of the problem by implementing an implicit Euler scheme for the discretization of the time derivative. We prove that the solution of the resulting system converges in the space $C([0, T], L^2(\mathbb{R}))$ to the solution of the finite system of ordinary differential equations approximating the original equation.

Finally, some numerical simulations are provided in the last section.

2. Previous results

In the previous paper [8] the authors considered the convergence of finite-difference approximations of the problem

$$\frac{\partial p}{\partial t} - D(p(t))\frac{\partial^2 p}{\partial \sigma^2} + \frac{1}{T_0}\chi_{\mathbb{R}\setminus[-1,1]}(\sigma)p = \frac{D(p(t))}{\alpha}\delta_0(\sigma), \tag{1}$$

$$p \geq 0, \quad p(0,\sigma) = p^0(\sigma), \tag{2}$$

where $p = p(t,\sigma)$, $t \in [0,T]$, $\sigma \in \mathbb{R}$, T_0 and α are positive constants.

Here, δ_0 is the Dirac δ-function with support in the origin,

$$D(p(t)) = \frac{\alpha}{T_0}\int_{|\sigma|>1} p(t,\sigma)\,d\sigma$$

and χ_I is the indicator function in the interval I.

The function $p(t,\sigma)$ is a probability density at time t, so for any $t \in [0,T]$,

$$\int_{\mathbb{R}} p(t,\sigma)d\sigma = 1, \qquad p(t,\sigma) \geq 0, \quad \text{for a.a. } \sigma \in \mathbb{R}. \tag{3}$$

It is well-known [2] that for any $p^0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $p^0 \geq 0$ a.e., $\int_{\mathbb{R}} p^0(\sigma)d\sigma = 1$, $\int_{\mathbb{R}} |\sigma|\, p^0(\sigma)d\sigma < \infty$ and $D(p_0) > 0$ there exists a unique solution $p = p(t,\sigma)$ of problem (1)–(2), which satisfies (3).

We consider as a first step the approximative problem

$$\partial_t p^c - \left(D\left(p^c(t)\right) + \frac{1}{c}\right)\partial^2_{\sigma\sigma}p + \frac{1}{T_0}\chi_{\mathbb{R}\setminus[-1,1]}(\sigma)p^c = \frac{D(p^c(t))}{\alpha}\delta_c(\sigma), \tag{4}$$

$$p^c \geq 0, \quad p^c(0,\sigma) = p^0_c(\sigma), \tag{5}$$

where $p^c = p^c(t,\sigma)$, $c > 0$ is a large parameter and the δ-function δ_0 is replaced by the step continuous from the right function

$$\delta_c(\sigma) = \begin{cases} 0, & \text{if } \sigma < -\frac{1}{2c}, \\ c, & \text{if } -\frac{1}{2c} \leq \sigma < \frac{1}{2c}, \\ 0, & \text{if } \sigma \geq \frac{1}{2c}. \end{cases}$$

We would like to highlight the fact that the new term $\frac{1}{c}\partial^2_{\sigma\sigma}p$ is an artificial diffusion which helps us to prove the convergence of the approximative solutions. Such a trick is very common in the numerical approximations of problems in Physics. Also, $[-\frac{1}{2c}, \frac{1}{2c}]$ is the support of the map δ_c, which approximates the δ-function δ_0. Therefore, when $c \to +\infty$, the artificial diffusion and the support of δ_c converge to 0 in unison.

Let p^0_c be such that

$$p^0_c \in C^\infty_0(\mathbb{R}), \quad p^0_c \geq 0 \text{ a.e.}, \int_{\mathbb{R}} p^0_c(\sigma)d\sigma = 1, \tag{6}$$

$$p^0_c \to p^0 \quad \text{in } L^2(\mathbb{R}), \qquad \sigma p^0_c \to \sigma p^0 \quad \text{in } L^1(\mathbb{R}), \text{ as } c \to +\infty. \tag{7}$$

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It is proved in [8, Theorem 3] that such approximation exists and that the unique solution p^c to problem (4)–(5) converges to the unique solution p to problem (1)–(2) in the space $C([0,T],X)$, where

$$X=\left\{p\in L^2(\mathbb{R}):\int_{\mathbb{R}}|\sigma|\,|p|\,d\sigma<+\infty\right\},$$

endowed with the norm $\|p\|_X=\|p\|_{L^2(\mathbb{R})}+\int_{\mathbb{R}}|\sigma|\,|p|\,d\sigma$. It is important to remark that for all $t\in[0,T]$ the solution satisfies $\int_{\mathbb{R}}p^c(t,\sigma)d\sigma=1$ and $p^c(t,\sigma)\geq 0$, for a.a. $\sigma\in\mathbb{R}$, since it is a probability density.

Further, we shall consider the following approximating infinite system of ordinary differential equations:

$$\frac{dp_i^{c,h}}{dt}-\left(D_h(p_h^c(t))+\frac{1}{c}\right)\frac{p_{i+1}^{c,h}(t)-2p_i^{c,h}(t)+p_{i-1}^{c,h}(t)}{h^2}+\frac{1}{T_0}\chi_{\mathbb{Z}\setminus[-2n_1,2n_1]}(i)\,p_i^{c,h}(t)=\frac{D_h\left(p_h^c(t)\right)}{\alpha}\delta_c^i, \tag{8}$$

$$p_i^{c,h}(0)=p_{c,h,i}^0,\quad i\in\mathbb{Z}, \tag{9}$$

where $h>0$ $\sigma_i=ih$, $i\in\mathbb{Z}$, $\frac{1}{h}=2n_1$, with $n_1\in\mathbb{N}$,

$$\chi_{\mathbb{Z}\setminus[-2n_1,2n_1]}(i)=\begin{cases}1, & \text{if } i\notin[-2n_1,2n_1],\\ 0, & \text{otherwise.}\end{cases}$$

Here, $p_h^c(t)=\left\{p_i^{c,h}(t)\right\}_{i\in\mathbb{Z}}$ denotes a sequence satisfying

$$p_i^{c,h}(t)\simeq p^c(t,\sigma_i),$$

and we made the following approximations

$$D\left(p^c(t)\right)=\frac{\alpha}{T_0}\int_{|\sigma|>1}p^c(t,\sigma)\,d\sigma\simeq D_h\left(p_h^c(t)\right)=\frac{\alpha h}{T_0}\sum_{|i|>2n_1}p_i^{c,h}(t), \tag{10}$$

$$\delta_c^i=\delta_c(\sigma_i)=\begin{cases}0, & \text{if } i<-n_{h,c},\\ c, & \text{if } -n_{h,c}\leq i<n_{h,c},\\ 0, & \text{if } i\geq n_{h,c}.\end{cases} \tag{11}$$

Also, c is taken such that $\frac{1}{2ch}=n_{h,c}\in\mathbb{N}$ and $n_{h,c}<2n_1$.

We observe that the parameter h is the length of the intervals in the finite-difference approximation of the second derivative $\partial^2_{\sigma\sigma}p$, which is a diffusion term. The approximation is getting better as h goes to 0. Also, $n_{h,c}$ is the number of subintervals of length h of the interval $[0,\frac{1}{2c}]$, the support of the function δ_c in the positive semi-axis. The condition $n_{h,c}<2n_1$ is equivalent to say that $\frac{1}{2c}<1$, that is, the support of δ_c is strictly included in the interval $[-1,1]$.

We define the following partition of the real line:

$$\Omega_h=\{\sigma_i=ih\}_{i\in\mathbb{Z}},\qquad I_i^h=[\sigma_i,\sigma_i+h).$$

For p_c^0 from (6) we define the step function

$$\widetilde{p}_{c,h}^0(\sigma)=\sum_{i\in\mathbb{Z}}p_c^0(ih)\,\chi_{I_i^h}(\sigma)$$

and normalize it by setting

$$p_{c,h}^0(\sigma)=\frac{\widetilde{p}_{c,h}^0(\sigma)}{\left\|\widetilde{p}_{c,h}^0\right\|_{L^1(\mathbb{R})}}. \tag{12}$$

It holds that $p_{c,h}^0\to p_c^0$ in $L^2(\mathbb{R})\cap L^1(\mathbb{R})$ as $h\to 0$.

Further, we fix $c\in\mathbb{N}$ and take a sequence $h_n\to 0$ such that $\frac{1}{2ch_n}=n_{h_n,c}\in\mathbb{N}$. Then conditions $\frac{1}{h_n}=2n_1$, $n_1\in\mathbb{N}$, $n_{h_n,c}<2n_1$ are satisfied. We take $p_{c,h_n,i}^0=p_{c,h_n}^0(ih_n)$ as the initial data in problem (8) and define the step functions

$$p_{h_n}^c(t,\sigma)=\sum_{i\in\mathbb{Z}}p_i^{c,h_n}(t)\,\chi_{I_i^h}(\sigma), \tag{13}$$

where $p_{h_n}^c(t)=\{p_i^{c,h_n}(t)\}_{i\in\mathbb{Z}}$ is the unique solution to problem (8)–(9). It is proved in [8, Theorem 2] that

$$p_{h_n}^c\to p^c\quad\text{strongly in } C([0,T];L^2(\mathbb{R})), \tag{14}$$

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where p^c is the solution to problem (4)–(5) with initial data p_c^0. As before, the solutions $p_{h_n}^c$ satisfy that

$$\int_{\mathbb{R}} p_{h_n}^c(t,\sigma)\,d\sigma = \sum_{i\in\mathbb{Z}} p_i^{c,h_n}(t)\,h_n = 1 \quad \text{and} \quad p_i^{c,h_n}(t) \geq 0, \quad \text{for any } t\in[0,T],\ i\in\mathbb{Z}.$$

Let us consider now finite-dimensional approximations. We define the operator $A_h^N : \mathbb{R}^{2N+1} \to \mathbb{R}^{2N+1}$ by

$$A_h^N := \frac{1}{h^2}\begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \ddots & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \ddots & -1 & 2 & -1 & 0 \\ 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}_{(2N+1)\times(2N+1)}. \tag{15}$$

Then we consider the finite-dimensional system

$$\begin{cases} \dfrac{dp_{h,i}^{c,N}}{dt} = -\left(D_h^N\left(p_h^{c,N}(t)\right) + \dfrac{1}{c}\right)\left(A_h^N p_h^{c,N}\right)_i - \dfrac{1}{T_0}\chi_{\mathbb{Z}\setminus[-2n_1,2n_1]}(i)\,p_{h,i}^{c,N} + \dfrac{D_h^N\left(p_h^{c,N}(t)\right)}{\alpha}\delta_c^i, \\ p_{h,i}^{c,N}(0) = p_{c,h,i}^{N,0}, \quad -N\leq i\leq N, \end{cases} \tag{16}$$

where $N > 2n_1$, $\frac{1}{h} = 2n_1$, $n_1\in\mathbb{N}$, $\frac{1}{2ch} = n_{h,c}\in\mathbb{N}$, $n_{h,c} < 2n_1$ and

$$D_h^N\left(p_h^{c,N}\right) = \frac{\alpha h}{T_0}\sum_{2n_1<|i|\leq N} p_{h,i}^{c,N}.$$

What we have done is to cut the tails of the system (8) off in order to work with a finite number of equations. The condition $N > 2n_1$ implies that we solve the problem for σ in an interval containing $[-1, 1]$. It is obvious that N has to be large to get good approximations.

For the initial data $p_{c,h}^0$ from (12) we consider the approximations $p_{c,h}^{N,0}$ given by

$$p_{c,h,i}^{N,0} = \frac{p_{c,h,i}^0}{\sum\limits_{|i|\leq N} hp_{c,h,i}^0}, \quad \text{for } |i|\leq N, \tag{17}$$

where $p_{c,h,i}^0 = p_{c,h}^0(ih)$. Then we define the step functions

$$p_h^{c,N}(t,\sigma) = \sum_{i\in\mathbb{Z}} p_{h,i}^{c,N}(t)\,\chi_{I_i}(\sigma), \tag{18}$$

where $\left\{p_{h,i}^{c,N}(\cdot)\right\}_{|i|\leq N}$ is the unique solution to problem (16) with initial data (17) and $p_{h,i}^{c,N}(t) = 0$ if $|i| > N$. It is proved in [8, Section 5] that

$$p_h^{c,N} \to p_h^c \quad \text{in } C([0,T], L^2(\mathbb{R})) \text{ as } N\to\infty,$$

where p_h^c is the function defined in (13) with initial data (12). Again, the property of being a probability density is satisfied:

$$\int_{\mathbb{R}} p_h^{c,N}(t,\sigma)\,d\sigma = \sum_{|i|\leq N} p_{h,i}^{c,N}(t)\,h = 1 \quad \text{and} \quad p_{h,i}^{c,N}(t)\geq 0, \quad \text{for any } t\geq 0,\ |i|\leq N.$$

3. Fixed points of approximations

Our aim in this section is to study the fixed points of the approximative problems and their convergence to the fixed points of the original problem (1). For simplicity, we shall consider the particular case where $T_0 = 1$.

First, recall that for Eq. (1) with $T_0 = 1$ the fixed points, given by the solutions of

$$D(p)\frac{\partial^2 p}{\partial\sigma^2} - \chi_{\mathbb{R}\setminus[-1,1]}(\sigma)\,p + \frac{D(p)}{\alpha}\delta_0(\sigma) = 0, \tag{19}$$

are the following [2]:

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- Any probability density $p(\cdot)$ with support in $[-1,1]$ solves (19). We note that all these solutions satisfy $D(p)=0$;
- If $\alpha \le \frac{1}{2}$, there are no more equilibria. If $\alpha > \frac{1}{2}$, then there exists a unique fixed point $\overline{p}$ with positive value of $D(\overline{p})$, which is given by

$$\overline{p}(\sigma) = \begin{cases} \dfrac{\sqrt{D^*}}{2\alpha} e^{(1+\sigma)/\sqrt{D^*}}, & \text{if } \sigma \le -1, \\ \dfrac{\sqrt{D^*}+1}{2\alpha} + \dfrac{1}{2\alpha}\sigma, & \text{if } -1 \le \sigma \le 0, \\ \dfrac{\sqrt{D^*}+1}{2\alpha} - \dfrac{1}{2\alpha}\sigma, & \text{if } 0 \le \sigma \le 1, \\ \dfrac{\sqrt{D^*}}{2\alpha} e^{(1-\sigma)/\sqrt{D^*}}, & \text{if } \sigma \ge 1, \end{cases} \tag{20}$$

where

$$D^* = \left(-\frac{1}{2} + \frac{\sqrt{4\alpha - 1}}{2} \right)^2 \tag{21}$$

and $z = \sqrt{D^*}$ is the unique positive solution of the equation

$$h(z) = z^2 + z - \alpha + \frac{1}{2} = 0.$$

We observe that when $\alpha > \frac{1}{2}$ the stationary point $\overline{p}$ is asymptotically stable [4]. Moreover, the numerical simulations in [7] suggest that every solution with initial data satisfying $D\left(p^0\right) > 0$ converges to this fixed point as time goes to $+\infty$.

We shall prove that for $\alpha > \frac{1}{2}$ the approximative problems possess a unique fixed point converging to (20).

3.1. Equation with large diffusion

Let us consider now the fixed points of problem (4). In order to find them we fix first $D > 0$ and solve first the following ordinary differential equation:

$$\left(D + \frac{1}{c} \right) \frac{d^2 p^c}{d\sigma^2} - \chi_{\mathbb{R} \setminus [-1,1]}(\sigma) p^c + \frac{D}{\alpha} \delta_c(\sigma) = 0.$$

We note that in this case, unlike problem (1), there is no stationary solutions with $D(p) = 0$.

Taking into account the condition $p^c(\sigma) \to 0$, as $\sigma \to \pm\infty$, it is not difficult to check that this equation possesses a unique solution defined by

$$p_D^c(\sigma) = \begin{cases} \dfrac{D}{2\alpha\sqrt{D+\frac{1}{c}}} e^{(1+\sigma)/\sqrt{D+\frac{1}{c}}}, & \text{if } \sigma \le -1, \\ \dfrac{D}{2\alpha} \dfrac{1+\sqrt{D+\frac{1}{c}}}{D+\frac{1}{c}} + \dfrac{D}{2\alpha\left(D+\frac{1}{c}\right)}\sigma, & \text{if } -1 \le \sigma \le -\dfrac{1}{2c}, \\ \dfrac{D}{2\alpha} \dfrac{1+\sqrt{D+\frac{1}{c}}}{D+\frac{1}{c}} - \dfrac{1}{8} \dfrac{D}{\alpha\left(D+\frac{1}{c}\right)c} - \dfrac{Dc}{2\alpha\left(D+\frac{1}{c}\right)}\sigma^2, & \text{if } -\dfrac{1}{2c} \le \sigma \le \dfrac{1}{2c}, \\ \dfrac{D}{2\alpha} \dfrac{1+\sqrt{D+\frac{1}{c}}}{D+\frac{1}{c}} - \dfrac{D}{2\alpha\left(D+\frac{1}{c}\right)}\sigma, & \text{if } \dfrac{1}{2c} \le \sigma \le 1, \\ \dfrac{D}{2\alpha\sqrt{D+\frac{1}{c}}} e^{(1-\sigma)/\sqrt{D+\frac{1}{c}}}, & \text{if } \sigma \ge 1. \end{cases}$$

Since

$$D\left(p_D^c\right) = \alpha \int_{|\sigma|>1} p_D^c(\sigma)\, d\sigma = D, \quad \text{for any } D > 0,$$

in order to obtain a fixed point it remains to find a positive value of D such that $\int_{\mathbb{R}} p_D^c(\sigma)\, d\sigma = 1$. Calculating the integral we obtain

$$\frac{1}{48c^2\alpha}\frac{2D}{D+\frac{1}{c}}\left(24c + 24c^2 D + 24c^2\sqrt{D+\frac{1}{C}} + 12c^2 - 1\right) = 1,$$

and after the change of variable $z = \sqrt{D+\frac{1}{c}}$ we finally have the equation

$$g^c(z) = z^4 + z^3 - \left(\frac{1}{24c^2} + \frac{1}{c} - \frac{1}{2} + \alpha\right)z^2 - \frac{1}{c}z - \frac{1}{2c} + \frac{1}{24c^3} = 0. \tag{22}$$

It follows from the Descarte's rule of signs that for $\alpha > 0.5$ and c large enough this polynomial possesses a unique positive root z^c. More precisely, c has to satisfy $c > \frac{1}{\sqrt{12}}$. We will take $c \geq 1$. Such condition is compatible with the meaning of the term $\frac{1}{c}\partial^2_{\sigma\sigma}p$ in (4), as this is an artificial diffusion that has to be small in order to approximate the original system properly, which means that we need c to be large.

If we pass to the limit as $c \to \infty$ the polynomial $g^c(z)$ tends to

$$g(z) = z^2\left(z^2 + z + \frac{1}{2} - \alpha\right).$$

By continuity, the root z^c converges to the unique positive root of $h(z)$, which is equal to $z^* = \sqrt{D^*}$. Therefore,

$$D^c = \left(z^c\right)^2 - \frac{1}{c} \to D^* > 0,$$

where D^* is given in (21). Hence, $D^c > 0$, for c large enough, and thus there is a unique stationary point $\overline{p}^c(\sigma) = p^c_{D^c}(\sigma)$. Moreover, it is easy to see using $D^c \to D^*$ that

$$\overline{p}^c \to \overline{p} \quad \text{in } X.$$

Therefore, we have proved the following result.

Theorem 1. *Let $\alpha > 0.5$. Problem (4) possesses a unique fixed point $\overline{p}^c$ for $c \geq 1$ and*

$$\overline{p}^c \to \overline{p} \quad \text{in } X, \text{ as } c \to \infty,$$

where $\overline{p}$ is the unique fixed point of problem (1) such that $D(\overline{p}) > 0$ defined in (20).

3.2. Lattice dynamical system

Further, we will study the fixed points of Eq. (8) with $\frac{1}{2ch} = n_{h,c} \in \mathbb{N}$ and $n_{h,c} < 2n_1 = \frac{1}{h}$. As before, we fix first $D > 0$ and solve the following equation in differences

$$\begin{cases} -\dfrac{D+\frac{1}{c}}{h^2}(p_{i+1} - 2p_i + p_{i-1}) + p_i = 0, & \text{if } i < -2n_1, \\ p_{i+1} - 2p_i + p_{i-1} = 0, & \text{if } -2n_1 \leq i < -n_{h,c}, \\ -\dfrac{D+\frac{1}{c}}{h^2}(p_{i+1} - 2p_i + p_{i-1}) = \dfrac{D}{\alpha}c, & \text{if } -n_{h,c} \leq i < n_{h,c}, \\ p_{i+1} - 2p_i + p_{i-1} = 0, & \text{if } n_{h,c} \leq i \leq 2n_1, \\ -\dfrac{D+\frac{1}{c}}{h^2}(p_{i+1} - 2p_i + p_{i-1}) + p_i = 0, & \text{if } i > 2n_1, \end{cases} \tag{23}$$

whose solution, taking into account that $p_i \underset{i\to\pm\infty}{\to} 0$, is given by

$$p^{c,h}_{i,D} = \begin{cases} C_1\lambda_1^i, & \text{if } i < -2n_1, \\ A + Bi, & \text{if } -2n_1 \leq i < -n_{h,c}, \\ E + Fi - \dfrac{ch^2 D}{2\left(D+\frac{1}{c}\right)\alpha}i^2, & \text{if } -n_{h,c} \leq i < n_{h,c}, \\ \overline{A} + \overline{B}i, & \text{if } n_{h,c} \leq i \leq 2n_1, \\ \overline{C}_2\lambda_1^{-i}, & \text{if } i > 2n_1, \end{cases} \tag{24}$$

provided that

$$\lambda_1=\lambda_{1,h}^{c,D}=1+\frac{h^2}{2\left(D+\frac{1}{c}\right)}+\frac{h}{\sqrt{D+\frac{1}{c}}}\sqrt{1+\frac{h^2}{4\left(D+\frac{1}{c}\right)}},\qquad \lambda_2=\lambda_{2,h}^{c,D}=\frac{1}{\lambda_{1,h}^{c,D}},\tag{25}$$

and the constants $C_1, A, B, E, F, \overline{A}, \overline{B}, \overline{C}_2$ satisfy the compatibility conditions

$$\begin{cases} A-\frac{1}{h}B-\lambda_1^{-\frac{1}{h}}C_1=0,\\ A-\frac{h+1}{h}B-\lambda_1^{-\frac{h+1}{h}}C_1=0,\\ A-\frac{1}{2ch}B-E+\frac{1}{2ch}F=-\frac{D}{8c\left(D+\frac{1}{c}\right)\alpha},\\ A-\frac{2ch+1}{2ch}B-E+\frac{2ch+1}{2ch}F=-\frac{D\left(1+2ch\right)^2}{8c\left(D+\frac{1}{c}\right)\alpha},\\ \overline{A}+\frac{1-2ch}{2ch}\overline{B}-E-\frac{1-2ch}{2ch}F=-\frac{D\left(1-2ch\right)^2}{8c\left(D+\frac{1}{c}\right)\alpha},\\ \overline{A}+\frac{1}{2ch}\overline{B}-E-\frac{1}{2ch}F=-\frac{D}{8c\left(D+\frac{1}{c}\right)\alpha},\\ \overline{A}+\frac{h+1}{h}B-\lambda_1^{-\frac{h+1}{h}}\overline{C}_2=0,\\ \overline{A}+\frac{1}{h}B-\lambda_1^{-\frac{1}{h}}\overline{C}_2=0. \end{cases}\tag{26}$$

Solving this system we obtain:

$$C_1=\frac{\lambda_1^{\frac{1}{h}}D}{\left(D+\frac{1}{c}\right)\alpha}\frac{\left(-1+\lambda_1^{-1}\right)h\left(2+h\right)-2h^2}{4\left(-1+\lambda_1^{-1}\right)\left(1+h-\lambda_1^{-1}\right)},\qquad \overline{C}_2=\frac{\lambda_1^{\frac{1}{h}}D}{\left(D+\frac{1}{c}\right)\alpha}\frac{\left(-1+\lambda_1^{-1}\right)h\left(2-h\right)-2h^2}{4\left(-1+\lambda_1^{-1}\right)\left(1+h-\lambda_1^{-1}\right)},\tag{27}$$

$$A=\frac{D}{\left(D+\frac{1}{c}\right)\alpha}\frac{\left(-1+\lambda_1^{-1}\right)\left(2+h\right)-2h}{4\left(-1+\lambda_1^{-1}\right)},\qquad \overline{A}=\frac{D}{\left(D+\frac{1}{c}\right)\alpha}\frac{\left(-1+\lambda_1^{-1}\right)\left(2-h\right)-2h}{4\left(-1+\lambda_1^{-1}\right)},$$

$$B=\frac{-D}{\left(D+\frac{1}{c}\right)\alpha}\frac{\left(-1+\lambda_1^{-1}\right)h\left(2+h\right)-2h^2}{4\left(1+h-\lambda_1^{-1}\right)},\qquad \overline{B}=\frac{D}{\left(D+\frac{1}{c}\right)\alpha}\frac{\left(-1+\lambda_1^{-1}\right)h\left(2-h\right)-2h^2}{4\left(1+h-\lambda_1^{-1}\right)},$$

$$F=-\frac{Dh}{2\left(D+\frac{1}{c}\right)\alpha}\left(1+ch\right)-\frac{D}{\left(D+\frac{1}{c}\right)\alpha}\frac{\left(-1+\lambda_1^{-1}\right)h\left(2+h\right)-2h^2}{4\left(1+h-\lambda_1^{-1}\right)},$$

$$E=-\frac{D}{8c\left(D+\frac{1}{c}\right)\alpha}\left(1+2ch\right)+\frac{D}{\left(D+\frac{1}{c}\right)\alpha}\frac{\left(-1+\lambda_1^{-1}\right)\left(2+h\right)-2h}{4\left(-1+\lambda_1^{-1}\right)}.$$

For simplicity of notation here and throughout the paper, if no confusion is possible, sometimes we omit the indexes c, h, D and write just λ_1.

We need to check first that $\alpha h\sum_{|i|>2n_1}p_{i,D}^{c,h}=D$. Indeed, we can easily compute that

$$\alpha h\sum_{|i|>2n_1}p_{i,D}^{c,h}=\frac{Dh^2}{\left(D+\frac{1}{c}\right)}\frac{\lambda_1}{\left(\lambda_1-1\right)^2}$$

$$=D\frac{4\left(D+\frac{1}{c}\right)+2h^2+2h\sqrt{4\left(D+\frac{1}{c}\right)+h^2}}{\left(h+\sqrt{4\left(D+\frac{1}{c}\right)+h^2}\right)^2}=D\quad\text{for any } D>0.$$

We need to find $D_h^c>0$ such that $S_h^c=\sum_{i\in\mathbb{Z}}p_{i,D_h^c}^{c,h}h=1$. Using mathematical software we obtain

$$S_h^c\left(D\right)=\sum_{i\in\mathbb{Z}}p_{i,D}^{c,h}h=\frac{b_h^c\left(D\right)}{w_h^c\left(D\right)},\tag{28}$$

where

$$b_h^c(D) = D\left(-1-2c^2\left(-6+h^2\right)\right) + 2D\left(1+4c^2\left(-3-3h+2h^2\right)\right)\lambda_{1,h}^{c,D} + D\left(-1+2c^2\left(6+12h+5h^2\right)\right)\left(\lambda_{1,h}^{c,D}\right)^2,$$

$$w_h^c(D) = 24c(1+cD)\alpha\left(-1+\lambda_{1,h}^{c,D}\right)^2.$$

With the change of variable $D = -\frac{1}{c} - \frac{h^2}{4} + z^2$ we derive first

$$\sum_{i\in\mathbb{Z}} p_{i,D}^{c,h} h = I_h^c(z) = -\frac{\left(4+ch^2-4cz^2\right)\left(1-12c^2-12c^2h+2c^2h^2-24c^2z-12c^2hz-24c^2z^2\right)}{24c^3(h-2z)(h+2z)\alpha} = 1, \quad (29)$$

and then the equation

$$g_h^c(z) = z^4 + \left(1+\frac{h}{2}\right)z^3 - \left(\frac{1}{24c^2}+\frac{1}{c}-\frac{1}{2}-\frac{h}{2}+\frac{h^2}{3}+\alpha\right)z^2 - \left(\frac{1}{c}+\frac{h}{2c}+\frac{h^2}{4}+\frac{h^3}{8}\right)z - \frac{1}{2c} + \frac{1}{24c^3} - \frac{h}{2c} + \frac{h^2}{96c^2} + \frac{h^2}{12c} - \frac{h^2}{8} - \frac{h^3}{8} + \frac{h^2\alpha}{4} + \frac{h^4}{48} = 0.$$

Again, it follows from the Descarte's rule of signs that for $\alpha > 0.5$, $c \geq 1$ and h small enough this polynomial possesses a unique positive root z_h^c. Moreover, since $g_h^c \to g^c$, as $h \to 0$, we have

$$z_h^c \to z^c,$$

where z^c is the unique positive solution to (22). Therefore,

$$D_h^c = -\frac{1}{c} - \frac{h^2}{4} + \left(z_h^c\right)^2 \to -\frac{1}{c} + \left(z^c\right)^2 = D^c > 0, \quad \text{as } h \to 0.$$

Hence, $D_h^c > 0$ for h small enough. More precisely, h has to satisfy the following conditions:

$$\frac{1}{24c^2}+\frac{1}{c}-\frac{1}{2}-\frac{h}{2}+\frac{h^2}{3}+\alpha > 0,$$
$$-\frac{1}{2c} + \frac{1}{24c^3} - \frac{h}{2c} + \frac{h^2}{96c^2} + \frac{h^2}{12c} - \frac{h^2}{8} - \frac{h^3}{8} + \frac{h^2\alpha}{4} + \frac{h^4}{48} < 0.$$

These conditions are satisfied for h small enough. This is compatible with the mechanical meaning of this parameter as described in Section 2. We can also draw some conclusions about the relationship between α, c and h. From the first inequality it is easily deduced that if $\alpha \to \frac{1}{2}$ and $c \to +\infty$, then we need that $h \to 0$. However, if c is fixed and $\alpha \to \frac{1}{2}$, then h does not need to go to 0. On the other hand, from the second inequality, since $\frac{h^2\alpha}{4} - \frac{h^2}{8} > 0$, it follows that when $c \to +\infty$ or $\alpha \to +\infty$, then $h \to 0$. Therefore, we obtain the following implications:

1. When c increases, the parameter h decreases.
2. When α increases, the parameter h decreases.

On the other hand, we can see that the function $I_h^c(z)$ defined in (29) is strictly increasing for $z > 0, z \neq \frac{h}{2}$. This follows from the facts that the polynomial $p(z) = 1-12c^2-12c^2h+2c^2h^2-24c^2z-12c^2hz-24c^2z^2$ is strictly decreasing for $z > 0$ and that the rational function $r(z) = \frac{4+c\left(h^2-4z^2\right)}{h^2-4z^2}$ is strictly increasing when $z > 0, z \neq \frac{h}{2}$. Also, since $I_h^c(z) \to +\infty$, as $z \to +\infty$, and $I_h^c(z) \to -\infty$, as $z \to \left(\frac{h}{2}\right)^+$, it is clear that $z_h^c > \frac{h}{2}$. In particular, we also deduce that the function $D \mapsto S_h^c(D) = I_h^c\left(\sqrt{D+\frac{1}{c}+\frac{h^2}{4}}\right)$ is strictly increasing for $D > 0$.

We have obtained then the unique stationary solution of (8), given by $\overline{p}_h^c = \left\{p_{i,D_h^c}^{c,h}\right\}_{i\in\mathbb{Z}}$. As before, we define then the step function

$$\overline{p}_h^c(\sigma) = \sum_{i\in\mathbb{Z}} p_{i,D_h^c}^{c,h}\chi_{I_i^h}(\sigma).$$

We will prove that

$$\overline{p}_h^c \to \overline{p}^c \quad \text{in } L^2(\mathbb{R}). \quad (30)$$

We replace in (25) and (27) the value of D by D_h^c and add the indexes c, h to the notation of these constants (e.g. A will be now A_h^c). We note that $\lambda_{1,h}^c = \lambda_{1,h}^{c,D_h^c}$. Then it is not difficult to check that when $h \to 0$ the following convergences are true:

$$\frac{\lambda_{1,h}^c - 1}{h} \to \frac{1}{\sqrt{D^c + \frac{1}{c}}}, \qquad \left(\lambda_{1,h}^c\right)^{\frac{1}{h}} \to e^{\frac{1}{\sqrt{D^c+\frac{1}{c}}}}, \tag{31}$$

$$C_{1,h}^c, \overline{C}_{2,h}^c \to \frac{D^c}{2\left(D^c + \frac{1}{c}\right)\alpha} e^{\frac{1}{\sqrt{D+\frac{1}{c}}}},$$

$$A_h^c, \overline{A}_h^c \to \frac{D^c\left(1 + \sqrt{D^c + \frac{1}{c}}\right)}{2\left(D^c + \frac{1}{c}\right)\alpha},$$

$$\frac{B_h^c}{h} \to \frac{D^c}{2\left(D^c + \frac{1}{c}\right)\alpha}, \qquad \frac{\overline{B}_h^c}{h} \to -\frac{D^c}{2\left(D^c + \frac{1}{c}\right)\alpha},$$

$$E_h^c \to \frac{D^c\left(1 + \sqrt{D^c + \frac{1}{c}}\right)}{2\alpha\left(D^c + \frac{1}{c}\right)} - \frac{D^c}{8\alpha\left(D^c + \frac{1}{c}\right)c},$$

$$\frac{F_h^c}{h} \to 0.$$

Thus, it follows that

$$\overline{p}_h^c(\sigma) \to \overline{p}^c(\sigma) \quad \text{for any } \sigma \in \mathbb{R}. \tag{32}$$

Let us estimate the tails of $\overline{p}_h^c$ in the norm of the space $L^2(\mathbb{R})$. Let $\varepsilon > 0$ be arbitrary. We will prove the existence of h_0 and $T(\varepsilon) > 0$ such that

$$\int_{T(\varepsilon)}^{\infty} \left(\overline{p}_h^c(\sigma)\right)^2 d\sigma < \varepsilon \quad \text{for all } h \le h_0. \tag{33}$$

For any $T > 0$ we have that

$$\int_T^{\infty} \left(\overline{p}_h^c(\sigma)\right)^2 d\sigma \le \sum_{i=K(T,h)}^{\infty} \left(\overline{C}_{2,h}^c\right)^2 \left(\lambda_{1,h}^c\right)^{-2i} h = \left(\overline{C}_{2,h}^c\right)^2 h \left(\lambda_{1,h}^c\right)^{-2K(T,h)} \frac{1}{1 - \left(\lambda_{1,h}^c\right)^{-2}}$$

$$= \left(\overline{C}_{2,h}^c\right)^2 \frac{h}{\lambda_{1,h}^c - 1} \left(\lambda_{1,h}^c\right)^{-2K(T,h)} \frac{\left(\lambda_{1,h}^c\right)^2}{1 + \lambda_{1,h}^c},$$

where $K(T,h) \in \mathbb{N}$ is such that $T - h < K(T,h)h \le T$. Then (31) implies that there exist $C, h_0 > 0$ such that

$$\int_T^{\infty} \left(\overline{p}_h^c(\sigma)\right)^2 d\sigma \le C \left(\lambda_{1,h}^c\right)^{\frac{-2(T-h)}{h}} \quad \text{for all } h \le h_0.$$

Since

$$\left(\lambda_{1,h}^c\right)^{\frac{-2(T-h)}{h}} \to e^{-\frac{2T}{\sqrt{D^c+\frac{1}{c}}}}, \quad \text{as } h \to 0,$$

there are $T(\varepsilon) > 1$, $h_0 > 0$ such that $C\left(\lambda_{1,h}^c\right)^{\frac{-2(T(\varepsilon)-h)}{h}} \le \varepsilon$ for all $h \le h_0$. Therefore, (33) holds, and one can choose h_0, $T(\varepsilon)$ such that the same result is true for the integral $\int_{-\infty}^{-T(\varepsilon)} \left(\overline{p}_h^c(\sigma)\right)^2 d\sigma$.

Now let us take the norm of the space $L^2(-T(\varepsilon), T(\varepsilon))$. We observe that by (31) there exists a constant $R^c(T)$ such that for all $h \le h_0$ it holds

$$\left(C_{1,h}^c \left(\lambda_{1,h}^c\right)^i\right)^2 \le \left(C_{1,h}^c\right)^2 \left(\lambda_{1,h}^c\right)^{-\frac{2}{h}} \le R^c(T), \quad \text{if } -K(T(\varepsilon),h) \quad -1 \le i < -2n_1,$$

$$\left(A_h^c + B_h^c i\right)^2 \le 2\left(A_h^c\right)^2 + 2\frac{\left(B_h^c\right)^2}{h^2} T^2(\varepsilon) \le R^c(T), \quad \text{if } -2n_1 \le i < -n_{h,c},$$

$$\left(E_h^c + F_h^c i - \frac{ch^2 D_h^c}{2\left(D_h^c + \frac{1}{c}\right)\alpha} i^2\right)^2 \leq 3\left(E_h^c\right)^2 + 3\frac{\left(F_h^c\right)^2}{h^2} T^2(\varepsilon) + 3\frac{c^2\left(D_h^c\right)^2}{4\left(D_h^c + \frac{1}{c}\right)^2 \alpha^2} T^4(\varepsilon)$$
$$\leq R^c(T), \quad \text{if } -n_{h,c} \leq i < n_{h,c},$$

$$\left(\overline{A}_h^c + \overline{B}_h^c i\right)^2 \leq 2\left(\overline{A}_h^c\right)^2 + 2\frac{\left(\overline{B}_h^c\right)^2}{h^2} T^2(\varepsilon) \leq R^c(T), \quad \text{if } n_{h,c} \leq i \leq 2n_1,$$

$$\left(\overline{C}_{2,h}^c \left(\lambda_{1,h}^c\right)^{-i}\right)^2 \leq \left(\overline{C}_{2,h}^c\right)^2 \left(\lambda_{1,h}^c\right)^{-\frac{2}{h}} \leq R^c(T), \quad \text{if } 2n_1 < i \leq K(T(\varepsilon), h) + 1.$$

Therefore, $\left(\overline{p}_h^c(\sigma)\right)^2 \leq R^c(T)$ for all $\sigma \in [-T(\varepsilon), T(\varepsilon)]$. Using this estimate, (32) and Lebesgue's theorem we obtain

$$\overline{p}_h^c \to \overline{p}^c \quad \text{in } L^2(-T(\varepsilon), T(\varepsilon)). \tag{34}$$

We can estimate the tails of $\overline{p}_h^c$ in the norm of the space $L^1(\mathbb{R})$ in a similar way, proving therefore that

$$\overline{p}_h^c \to \overline{p}^c \quad \text{in } L^1(\mathbb{R}). \tag{35}$$

Finally, combining (34) and (33) we can prove in a standard way that (30) is true.

Summing up the results we have the following theorem.

Theorem 2. *Let $\alpha > 0.5$ and $\frac{1}{2ch} = n_{h,c} \in \mathbb{N}$, $n_{h,c} < 2n_1 = \frac{1}{h}$. If $c \geq 1$ and h is small enough, then problem (8) possesses a unique fixed point $\overline{p}_h^c$ and*

$$\overline{p}_h^c \to \overline{p}^c \quad \text{in } L^2(\mathbb{R}) \cap L^1(\mathbb{R}), \text{ as } h \to 0,$$

where $\overline{p}^c$ is the unique fixed point of problem (4).

3.3. Finite-dimensional approximations

The last step consists in studying the fixed points of problem (16) for h, c fixed and satisfying $\frac{1}{2ch} = n_{h,c} \in \mathbb{N}$, $n_{h,c} < 2n_1 = \frac{1}{h}$. Assume that $N > 2n_1$.

In the same way as before, we fix first $D > 0$ and solve the following difference equation

$$\begin{cases} -\dfrac{D + \frac{1}{c}}{h^2}\left(p_{i+1} - 2p_i + p_{i-1}\right) + p_i = 0, & \text{if } -N \leq i < -2n_1, \\ p_{i+1} - 2p_i + p_{i-1} = 0, & \text{if } -2n_1 \leq i < -n_{h,c}, \\ -\dfrac{D + \frac{1}{c}}{h^2}\left(p_{i+1} - 2p_i + p_{i-1}\right) = \dfrac{D}{\alpha} c, & \text{if } -n_{h,c} \leq i < n_{h,c}, \\ p_{i+1} - 2p_i + p_{i-1} = 0, & \text{if } n_{h,c} \leq i \leq 2n_1, \\ -\dfrac{D + \frac{1}{c}}{h^2}\left(p_{i+1} - 2p_i + p_{i-1}\right) + p_i = 0, & \text{if } -2n_1 < i \leq N, \end{cases} \tag{36}$$

whose solution is defined by

$$p_{i,D}^{c,h,N} = \begin{cases} C_1 \lambda_1^i + C_2 \lambda_1^{-i}, & \text{if } -N \leq i < -2n_1, \\ A + Bi, & \text{if } -2n_1 \leq i < -n_{h,c}, \\ E + Fi - \dfrac{ch^2 D}{2\left(D + \frac{1}{c}\right)\alpha} i^2, & \text{if } -n_{h,c} \leq i < n_{h,c}, \\ \overline{A} + \overline{B} i, & \text{if } n_{h,c} \leq i \leq 2n_1, \\ \overline{C}_1 \lambda_1^i + \overline{C}_2 \lambda_1^{-i}, & \text{if } 2n_1 < i \leq N. \end{cases} \tag{37}$$

The compatibility conditions in this case are the following:

$$\begin{cases} \left(\dfrac{\lambda_1^{-2N}-\lambda_1^{-2N-1}}{1-\lambda_1}\right)C_1+C_2=0, \\ A-\dfrac{1}{h}B-\lambda_1^{-\frac{1}{h}}C_1-\lambda_1^{\frac{1}{h}}C_2=0, \\ A-\dfrac{h+1}{h}B-\lambda_1^{-\frac{h+1}{h}}C_1-\lambda_1^{\frac{h+1}{h}}C_2=0, \\ A-\dfrac{1}{2ch}B-E+\dfrac{1}{2ch}F=-\dfrac{D}{8c\left(D+\frac{1}{c}\right)\alpha}, \\ A-\dfrac{2ch+1}{2ch}B-E+\dfrac{2ch+1}{2ch}F=-\dfrac{D\left(1+2ch\right)^2}{8c\left(D+\frac{1}{c}\right)\alpha}, \\ \overline{A}+\dfrac{1-2ch}{2ch}\overline{B}-E-\dfrac{1-2ch}{2ch}F=-\dfrac{D\left(1-2ch\right)^2}{8c\left(D+\frac{1}{c}\right)\alpha}, \\ \overline{A}+\dfrac{1}{2ch}\overline{B}-E-\dfrac{1}{2ch}F=-\dfrac{D}{8c\left(D+\frac{1}{c}\right)\alpha}, \\ \overline{A}+\dfrac{h+1}{h}\overline{B}-\lambda_1^{\frac{h+1}{h}}\overline{C}_1-\lambda_1^{-\frac{h+1}{h}}\overline{C}_2=0, \\ \overline{A}+\dfrac{1}{h}\overline{B}-\lambda_1^{\frac{1}{h}}\overline{C}_1-\lambda_1^{-\frac{1}{h}}\overline{C}_2=0, \\ \overline{C}_1+\left(\dfrac{\lambda_1^{-2N}-\lambda_1^{-2N-1}}{1-\lambda_1}\right)\overline{C}_2=0. \end{cases} \tag{38}$$

If we add to system (26) the equations

$$C_2=0,$$
$$\overline{C}_1=0,$$

and the same terms containing the constants $C_2, \overline{C}_1$ in the first two and the last two equations, then we obtain a new system which shares the same solution with system (26). We have therefore the same number of variables and equations in system (38) and the modified system (26). Moreover, the coefficients in system (38) converge to the corresponding ones in system (26) when $N\to\infty$. Writing systems (26), (38) in the matrix form

$$M_{h,D}^c x_{h,D}^c=b_{h,D}^c,$$
$$M_{h,D}^{c,N} x_{h,D}^{c,N}=b_{h,D}^c,$$

and noting that $M_{h,D}^c$, $M_{h,D}^{c,N}$ are invertible for N large enough, we have

$$x_{h,D}^{c,N}=\left(M_{h,D}^{c,N}\right)^{-1}b_{h,D}^c\to\left(M_{h,D}^c\right)^{-1}b_{h,D}^c=x_{h,D}^c \quad \text{in } \mathbb{R}^{10}. \tag{39}$$

For any $D^N>0$ we define then the step function

$$\overline{p}_h^{c,N,D^N}(\sigma)=\sum_{i\in\mathbb{Z}}p_{i,D^N}^{c,h,N}\chi_{I_i^h}(\sigma),$$

where $p_{i,D^N}^{c,h,N}=0$ if $|i|>N$. We will prove that if $D^N\to D$, then

$$\overline{p}_h^{c,N,D^N}\to\overline{p}_h^{c,D} \quad \text{in } L^2(\mathbb{R})\cap L^1(\mathbb{R}), \tag{40}$$

where $\overline{p}_h^{c,D}=\sum_{i\in\mathbb{Z}}p_{i,D}^{c,h}\chi_{I_i^h}(\sigma)$ is the step function which is defined from the solution of system (23) given by (24). For this let us estimate the tails of this summatory. For $K>2n_1$, $K\in\mathbb{N}$, we have that

$$\int_{Kh}^{+\infty}\overline{p}_h^{c,N,D^N}(\sigma)\,d\sigma=\sum_{i\geq K}p_{i,D^N}^{c,h,N}h=\sum_{i\geq K}\left(\overline{C}_{1,h,D^N}^{c,N}\left(\lambda_{1,h}^{c,D^N}\right)^i+\overline{C}_{2,h,D^N}^{c,N}\left(\lambda_{1,h}^{c,D^N}\right)^{-i}\right)h.$$

We note that $x_{h,D^N}^{c,N}\to x_{h,D}^c$ is also true. Hence, for any $\epsilon>0$ there exists $K(\epsilon)>2n_1$ such that

$$\sum_{i\geq K(\epsilon)}\overline{C}_{2,h,D^N}^{c,N}\left(\lambda_{1,h}^{c,D^N}\right)^{-i}h<\frac{\epsilon}{8}, \quad \text{for any } N>K(\epsilon).$$

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On the other hand, we have that

$$\sum_{i\geq K(\epsilon)} \overline{C}^{c,N}_{1,h,D^N}\left(\lambda^{c,D^N}_{1,h}\right)^i h = \overline{C}^{c,N}_{1,h,D^N}\frac{h}{\lambda^{c,D^N}_{1,h}-1}\left(\left(\lambda^{c,D^N}_{1,h}\right)^{N+1}-\left(\lambda^{c,D^N}_{1,h}\right)^{K(\epsilon)}\right)$$
$$\leq \overline{C}^{c,N}_{1,h,D^N}\left(\lambda^{c,D^N}_{1,h}\right)^{N+1}\frac{h}{\lambda^{c,D^N}_{1,h}-1}.$$

Using mathematical software it can be calculated that

$$\overline{C}^{c,N}_{1,h,D^N} = \frac{r^{c,N}_{h,D^N}}{l^{c,N}_{h,D^N}},$$

where

$$r^{c,N}_{h,D^N} = -cD^N h\left(\lambda^{c,D^N}_{1,h}\right)^{\frac{1}{h}}\left((2+h)\left(\lambda^{c,D^N}_{1,h}\right)^{2/h}+(-2+h)\left(\lambda^{c,D^N}_{1,h}\right)^{\frac{2+h}{h}}\right.$$
$$\left.+(-2+h)\left(\lambda^{c,D^N}_{1,h}\right)^{2N}+(2+h)\left(\lambda^{c,D^N}_{1,h}\right)^{1+2N}\right),$$
$$l^{c,N}_{h,D^N} = 4(1+cD^N)\alpha\left(-1+\lambda^{c,D^N}_{1,h}\right)\left(\left(\lambda^{c,D^N}_{1,h}\right)^{2/h}-\left(\lambda^{c,D^N}_{1,h}\right)^{2N}\right)\left((1+h)\left(\lambda^{c,D^N}_{1,h}\right)^{2/h}\right.$$
$$\left.-\left(\lambda^{c,D^N}_{1,h}\right)^{\frac{2+h}{h}}-\left(\lambda^{c,D^N}_{1,h}\right)^{2N}+(1+h)\left(\lambda^{c,D^N}_{1,h}\right)^{1+2N}\right).$$

Thus,

$$\overline{C}^{c,N}_{1,h,D^N}\leq L\left(c,h,D^N\right)\left(\lambda^{c,D^N}_{1,h}\right)^{-2N}\quad\text{for all } N,$$

where $L\left(c,h,D^N\right)$ is bounded as $D^N\to D$. Thus, using the formula for $\lambda^{c,D^N}_{1,h}$ given in (25) we can check that there exists $N(\epsilon)>0$ such that

$$\sum_{i\geq K(\epsilon)} \overline{C}^{c,N}_{1,h,D^N}\left(\lambda^{c,D^N}_{1,h}\right)^i h<\frac{\epsilon}{8}\quad\text{for all } N>N(\epsilon).$$

Arguing in the same way for negative i, we obtain the existence of $K(\epsilon)>0$ and $N(\epsilon)>K(\epsilon)$ such that

$$\int_{-\infty}^{-K(\epsilon)h}\overline{p}^{c,N,D^N}_h(\sigma)\,d\sigma+\int_{K(\epsilon)h}^{+\infty}\overline{p}^{c,N,D^N}_h(\sigma)\,d\sigma<\frac{\epsilon}{2},\quad\text{for any } N>N(\epsilon).$$

From (39) it follows also the existence of $N_1(\epsilon)$ such that

$$\int_{-K(\epsilon)}^{K(\epsilon)}\left|\overline{p}^{c,N,D^N}_h(\sigma)-\overline{p}^{c,N,D}_h(\sigma)\right|d\sigma<\frac{\epsilon}{2},\quad\text{for any } N>N_1(\epsilon).$$

Hence, $\overline{p}^{c,N,D^N}_h\to\overline{p}^{c,D}_h$ in $L^1(\mathbb{R})$. The convergence in the space $L^2(\mathbb{R})$ is proved in a similar way.

Using again mathematical software we check that

$$\alpha h\sum_{2n_1<|i|\leq N} p^{c,h,N}_{i,D^{c,N}_h}=D\quad\text{for any } D>0. \tag{41}$$

In order to find a stationary point of (16) it remains to show the existence of $D^{c,N}_h>0$ such that $S^{c,N}_h(D)=\sum_{|i|\leq N}p^{c,h,N}_{i,D^{c,N}_h}h=1$. In view of $\overline{p}^{c,N,D^N}_h\to\overline{p}^{c,D}_h$ in $L^1(\mathbb{R})$, if $D^N\to D$, we have that

$$\sum_{|i|\leq N}p^{c,h,N}_{i,D^N}h\to\sum_{i\in\mathbb{Z}}p^{c,h}_{i,D}h\quad\text{as } N\to\infty.$$

Since $\sum_{i\in\mathbb{Z}}p^{c,h}_{i,D^c_h}h=1$ and the function $D\mapsto S^c_h(D)=\sum_{i\in\mathbb{Z}}p^{c,h}_{i,D}h$ is strictly increasing for $D>0$ (see Section 3.2), one can find two values $D_1<D^c_h<D_2$ satisfying

$$\sum_{i\in\mathbb{Z}}p^{c,h}_{i,D_1}h<1<\sum_{i\in\mathbb{Z}}p^{c,h}_{i,D_2}h.$$

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Then there exists N_0 for which

$$\sum_{|i|\leq N} p_{i,D_1}^{c,h,N} h < 1 < \sum_{|i|\leq N} p_{i,D_2}^{c,h,N} h \quad \text{if } N \geq N_0.$$

If we prove that the map $D \mapsto \sum_{|i|\leq N} p_{i,D}^{c,h,N} h$ is continuous, then the existence of $D_h^{c,N} > 0$ such that $\sum_{i\in\mathbb{Z}} p_{i,D_h^{c,N}}^{c,h,N} h = 1$ follows. Indeed, as the matrix $M_{h,D}^{c,N}$ possesses a positive determinant and the coefficients of the system (38) depend continuously on $D > 0$, it follows that the vector of solutions of (38), $x_{h,D}^{c,N} = \left(M_{h,D}^{c,N}\right)^{-1} b_{h,D}^{c}$, depends continuously on $D > 0$ as well, and this implies easily the continuity of $D \mapsto \sum_{|i|\leq N} p_{i,D}^{c,h,N} h$. Therefore, for any $N \geq N_0$ there is at least one stationary point of (16), given by $\overline{p}_h^{c,N} = \left\{ p_{i,D_h^{c,N}}^{c,h,N} \right\}_{i\in\mathbb{Z}}$. As before, we define then the step function

$$\overline{p}_h^{c,N}(\sigma) = \sum_{i\in\mathbb{Z}} p_{i,D_h^{c,N}}^{c,h,N} \chi_{I_i^h}(\sigma),$$

where $p_{i,D_h^{c,N}}^{c,h,N} = 0$ if $|i| > N$. It is clear that $D_h^{c,N} \to D_h^c$. Indeed, suppose that there is a subsequence such that $D_h^{c,N_j} \to D^* \neq D_h^c$. Then (40) implies that $1 = S_h^{c,N}\left(D_h^{c,N_j}\right) \to S_h^c(D^*) \neq 1$, which is a contradiction.

Thus, in view of (40), $\overline{p}_h^{c,N}$ converges in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ to the unique stationary point of (8), $\overline{p}_h^c$. Moreover, assume that there exists another sequence $\widetilde{D}_h^{c,N} > 0$ such that $S_h^{c,N}\left(\widetilde{D}_h^{c,N}\right) = 1$. In view of (41) we have $\widetilde{D}_h^{c,N} \leq \alpha$ for any N. Thus $\widetilde{D}_h^{c,N} \to D_h^c$ and the corresponding stationary solution $\widetilde{p}_h^{c,N}$ converges to $\overline{p}_h^c$ in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ as well.

It remains to check that in fact $\overline{p}_h^{c,N}$ is the unique stationary point. Using mathematical software one can obtain that the value of the summatory $\sum_{|i|\leq N} p_{i,D}^{c,h,N}$ is given by

$$S_h^{c,N}(D) = \frac{f_h^{c,N}(D) + p_h^c(D)}{g_h^{c,N}(D) + q_h^c(D)},$$

where

$$p_h^c(D) = D(1 + 2c^2\left(-6 + h^2\right)) + 2D\left(-1 + 4c^2(3 + 3h - 2h^2)\right)\lambda_{1,h}^{c,D} - D\left(-1 + 2c^2(6 + 12h + 5h^2)\right)\left(\lambda_{1,h}^{c,D}\right)^2,$$

$$q_h^c(D) = -24c(1 + cD)\alpha\left(-1 + \lambda_{1,h}^{c,D}\right)^2,$$

and

$$f_h^{c,N}(D) = D\left(-1 + 2c^2(6 + h(12 + 5h))\right)\left(\lambda_{1,h}^{c,D}\right)^{\frac{2}{h} - 2N} - D\left(-2 + 8c^2(3 + (3 - 2h)h)\right)\left(\lambda_{1,h}^{c,D}\right)^{1 + \frac{2}{h} - 2N}$$
$$+ D\left(1 + 2c^2\left(-6 + h^2\right)\right)\left(\lambda_{1,h}^{c,D}\right)^{2 + \frac{2}{h} - 2N},$$

$$g_h^{c,N}(D) = 24c(1 + cD)\alpha\left(-1 + \lambda_{1,h}^{c,D}\right)^2\left(\lambda_{1,h}^{c,D}\right)^{\frac{2}{h} - 2N}.$$

It is easy to see that

$$\lim_{N\to\infty} S_h^{c,N}(D) = \frac{p_h^c(D)}{q_h^c(D)} = S_h^c(D),$$

uniformly in compact sets of $(0, \infty)$, where $S_h^c(D)$ is the summatory $\sum_{i\in\mathbb{Z}} p_{i,D}^{c,h} h$ in the previous approximation given in (28). Also, from

$$f_h^{c,N}(D), \frac{d}{dD} f_h^{c,N}(D) \to 0,$$
$$g_h^{c,N}(D), \frac{d}{dD} g_h^{c,N}(D) \to 0,$$

uniformly for D inside a compact set of $(0, \infty)$, and $q_h^c(D) > 0$ for any $D > 0$ we obtain that

$$\frac{d}{dD} S_h^{c,N}(D) \to \frac{d}{dD} S_h^c(D),$$

uniformly in compact sets of $(0, \infty)$. Let us consider an interval $[D_h^c - a, D_h^c + a]$ with $a > 0$. As $\frac{d}{dD} S_h^c(D) > 0$ for $D > 0$, there exists $N_1 > 0$ such that $\frac{d}{dD} S_h^{c,N}(D) > 0$, for all $D \in [D_h^c - a, D_h^c + a]$ and $N \geq N_1$, which implies that there can be

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only one value $D_h^{c,N} \in [D_h^c - a, D_h^c + a]$ satisfying $S_h^{c,N}\left(D_h^{c,N}\right) = 1$. But there cannot be a sequence $\widetilde{D}_h^{c,N} \notin [D_h^c - a, D_h^c + a]$ satisfying this property, because this would be in contradiction with $\widetilde{D}_h^{c,N} \to D_h^c$. Therefore, $D_h^{c,N}$ is unique and consequently so is $\overline{p}_h^{c,N}$.

We summarize the results of this section.

Theorem 3. *Let $\alpha > 0.5$ and $\frac{1}{2ch} = n_{h,c} \in \mathbb{N}$, $n_{h,c} < 2n_1 = \frac{1}{h}$. Assume that $N > 2n_1$. Let $c \geq 1$ and h be small enough. Problem (16) possesses a unique fixed point $\overline{p}_h^{c,N}$ for N large enough and*

$$\overline{p}_h^{c,N} \to \overline{p}_h^c \quad \text{in } L^2(\mathbb{R}) \cap L^1(\mathbb{R}), \text{ as } N \to \infty,$$

where $\overline{p}_h^c$ is the unique fixed point of problem (8).

3.4. Full convergence

Finally, we put together all the results of this section for a general perspective.

Let $\alpha > 0.5$. We take sequences $c_m \to \infty$, $c_m \in \mathbb{N}$, $h_{n,m} \to 0$ (as $n \to \infty$), such that $\frac{1}{2c_m h_n} = n_{h_n,c_m} \in \mathbb{N}$. Let $\overline{p}$, $\overline{p}^c$, $\overline{p}_h^c$, $\overline{p}_h^{c,N}$ be the fixed points of problems (1), (4), (8), (16), respectively. We have proved that

$$\begin{aligned}
&\overline{p}^{c_m} \to \overline{p} \quad \text{in } X, \text{ as } m \to \infty, \\
&\overline{p}_{h_{n,m}}^{c_m} \to \overline{p}^{c_m} \quad \text{in } L^2(\mathbb{R}) \cap L^1(\mathbb{R}), \text{ as } h_{n,m} \to 0, \\
&\overline{p}_{h_{n,m}}^{c_m,N} \to \overline{p}_{h_{n,m}}^{c_m} \quad \text{in } L^2(\mathbb{R}) \cap L^1(\mathbb{R}), \text{ as } N \to \infty.
\end{aligned}$$

Hence, the following iterate limit holds:

$$\lim_{m\to\infty} \lim_{n\to\infty} \lim_{N\to\infty} \overline{p}_{h_{n,m}}^{c_m,N} = \overline{p} \quad \text{in } L^2(\mathbb{R}) \cap L^1(\mathbb{R}).$$

4. An implicit Euler numerical method

In order to complete the sequence of approximations developed in Section 2 we shall discretize now system (16) with respect to the time variable using an implicit Euler scheme. Using the step $s > 0$ we consider the discrete moments of time $t_n = ns$, $n \geq 0$, and denote by $p_h^{c,N,n} = \{p_{h,i}^{c,N,n}\}_{|i|\leq N}$ the approximation of the solution $p_n^{c,N}(t)$ of problem (16) at $t = t_n$. For simplicity of notation we will write for the time being just p_i^n instead of $p_{h,i}^{c,N,n}$ for the components of the vector $p_h^{c,N,n}$. Then we obtain the following algebraic system:

$$\frac{p_i^{n+1} - p_i^n}{s} = -\left(D_h^N\left(p_h^{c,N,n}\right) + \frac{1}{c}\right)\left(A_h^N p_h^{c,N,n+1}\right)_i - \frac{1}{T_0}\chi_{\mathbb{Z}\setminus[-2n_1,2n_1]}(i)\, p_i^n + \frac{D_h^N\left(p_h^{c,N,n}\right)}{\alpha}\delta_c^i, \tag{42}$$

with initial data

$$p_{c,h}^{N,0} = \{p_i^0\}_{|i|\geq N} \in \mathbb{R}^{2N+1}. \tag{43}$$

System (42) can be rewritten as

$$\begin{aligned}
&-\left(D_h^N\left(p_h^{c,N,n}\right) + \frac{1}{c}\right)\left(d_1 p_{i+1}^{n+1} + d_2 p_{i-1}^{n+1}\right) + \left(\frac{h^2}{s} + d_3\left(D_h^N\left(p_h^{c,N,n}\right) + \frac{1}{c}\right)\right)p_i^{n+1} \\
&\quad = \frac{h^2}{s}p_i^n - \frac{h^2}{T_0}\chi_{\mathbb{Z}\setminus[-2n_1,2n_1]}(i)\, p_i^n + \frac{h^2 D_h^N\left(p_h^{c,N,n}\right)}{\alpha}\delta_c^i
\end{aligned}$$

where $d_1 = 1$ if $i \leq N-1$ and $d_1 = 0$ if $i = N$, whereas $d_2 = 1$ if $i \geq -N+1$ and $d_2 = 0$ if $i = -N$, and $d_3 = 2$ if $|i| \leq N-1$ and $d_3 = 1$ otherwise.

Denote $\beta = h^2/s$, $D^n = D_h^N\left(p_h^{c,N,n}\right) + \frac{1}{c}$, $T^n = 2D^n + \beta$, $\overline{T}^n = D^n + \beta$. Then we can express the system in the matrix form

$$M^n p_h^{c,N,n+1} = C^n p_h^{c,N,n} + \Delta^n, \tag{44}$$

where

$$M^n = \begin{pmatrix} \overline{T}^n & -D^n & 0 & \cdots & 0 & 0 \\ -D^n & T^n & -D^n & 0 & \cdots & 0 \\ 0 & -D^n & T^n & -D^n & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & \ddots & \ddots & T^n & -D^n \\ 0 & 0 & \cdots & 0 & -D^n & \overline{T}^n \end{pmatrix},$$

$\Delta_i^n = \frac{h^2 D_h^N\left(p_h^{c,N,n}\right)}{\alpha}\delta_c^i$ and C^n is a diagonal matrix such that

$$C_{ii}^n = \begin{cases} \frac{h^2}{s} - \frac{h^2}{T_0}, & \text{if } |i| > 2n_1, \\ \frac{h^2}{s}, & \text{if } |i| \le 2n_1. \end{cases}$$

The matrix M^n is invertible, so system (44) has a unique solution for any $n \ge 0$ and $p_h^{c,N,n} \in \mathbb{R}^{2N+1}$.

Denote by $\|\cdot\|_{\mathbb{R}^{2N+1}}$ and $(\cdot,\cdot)_{\mathbb{R}^{2N+1}}$ the usual norm and scalar product in the space $\mathbb{R}^{2N+1}$, in which we also consider the following equivalent norms:

$$\|p\|_{\bar{l}^{1N}} := \sum_{i=-N}^{N} (|i|+1)\,|p_i|\,,$$

$$\|p\|_{l_D^{1N}} := h \sum_{|i| \le N} |p_i|\,, \quad \text{for } p \in \mathbb{R}^{2N+1}.$$

First, we will define a semigroup with discrete time in the phase space

$$\mathbb{D}^N = \left\{ p \in \mathbb{R}^{2N+1} : p_i \ge 0,\ \|p\|_{l_D^{1N}} = 1 \right\}.$$

To this end we will establish some preliminary lemmas. Denote $v^+ = \max\{v, 0\}$, $v^- = \max\{-v, 0\}$ for $v \in \mathbb{R}$, and $p^+ = \left(p_{-N}^+, \ldots, p_N^+\right)$ for $p \in \mathbb{R}^{2N+1}$. Also, we introduce the matrix

$$B_h^N := \frac{1}{h}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}_{(2N+1)\times(2N+1)}, \tag{45}$$

which satisfies $A_h^N = \left(B_h^N\right)^t B_h^N$.

Lemma 4. *Assume that $s \le T_0$. If $p_i^n \ge 0$, for all i, then $p_i^{n+1} \ge 0$, for all i, as well.*

Proof. Since $D_h^N\left(p_h^{c,N,n}\right) \ge 0$, multiplying (42) by $\left(-p_h^{c,N,n+1}\right)^+$ and arguing in a similar way to [6, Lemma 4.1] we obtain

$$\left\| \left(-p_h^{c,N,n+1}\right)^+ \right\|_{\mathbb{R}^{2N+1}}^2 \le -s\left(D_h^N\left(p_h^{c,N,n}\right) + \frac{1}{c}\right)\left(A_h^N\left(-p_h^{c,N,n+1}\right), \left(-p_h^{c,N,n+1}\right)^+\right)_{\mathbb{R}^{2N+1}}. \tag{46}$$

We will check that $\left(A_h^N\left(-p_h^{c,N,n+1}\right), \left(-p_h^{c,N,n+1}\right)^+\right)_{\mathbb{R}^{2N+1}} \ge 0$. Indeed

$$\begin{aligned}\left(A_h^N\left(-p_h^{c,N,n+1}\right), \left(-p_h^{c,N,n+1}\right)^+\right)_{\mathbb{R}^{2N+1}} &= \left(A_h^N\left(\left(-p_h^{c,N,n+1}\right)^+ - \left(-p_h^{c,N,n+1}\right)^-\right), \left(-p_h^{c,N,n+1}\right)^+\right)_{\mathbb{R}^{2N+1}} \\ &= \left\|B_h^N\left(-p_h^{c,N,n+1}\right)^+\right\|_{\mathbb{R}^{2N+1}}^2 - \left(B_h^N\left(-p_h^{c,N,n+1}\right)^-, B_h^N\left(-p_h^{c,N,n+1}\right)^+\right)_{\mathbb{R}^{2N+1}},\end{aligned}$$

and the result follows from

$$\left(B_h^N\left(-p_h^{c,N,n+1}\right)^-, B_h^N\left(-p_h^{c,N,n+1}\right)^+\right)_{\mathbb{R}^{2N+1}} = \frac{1}{h^2}\sum_{i=-N}^{N-1}\left(\left(-p_{i+1}^{n+1}\right)^- - \left(-p_i^{n+1}\right)^-\right)\left(\left(-p_{i+1}^{n+1}\right)^+ - \left(-p_i^{n+1}\right)^+\right)$$
$$= -\frac{1}{h^2}\sum_{i=-N}^{N-1}\left(-p_i^{n+1}\right)^-\left(-p_{i+1}^{n+1}\right)^+ - \frac{1}{h^2}\sum_{i=-N}^{N-1}\left(-p_{i+1}^{n+1}\right)^-\left(-p_i^{n+1}\right)^+ \leq 0.$$

Therefore, we get from (46) that $\left\|\left(-p_h^{c,N,n+1}\right)^+\right\|^2_{\mathbb{R}^{2N+1}} \leq 0$. Hence $\left(-p_h^{c,N,n+1}\right)^+ = 0$, so $p_i^{n+1} \geq 0$ for all i. ■

Lemma 5. *Let $p_h^{c,N,n}$ be such that $\sum_{|i|\leq N} p_i^n h = 1$. Then $p_h^{c,N,n+1}$ satisfies $\sum_{|i|\leq N} p_i^{n+1} h = 1$.*

Proof. It is easy to see that the matrix A_h^N satisfies

$$\left(A_h^N u, v\right)_{\mathbb{R}^{2N+1}} = \left(u, A_h^N v\right)_{\mathbb{R}^{2N+1}} = 0, \quad \forall u \in \mathbb{R}^{2N+1}, \tag{47}$$

if $v = h\left(1, \ldots, 1, \overset{(i=0)}{1}, 1, \ldots, 1\right)$. Therefore, multiplying (42) by v we have

$$\sum_{|i|\leq N} \frac{p_i^{n+1} - p_i^n}{s} h = \frac{1}{T_0}\left(\sum_{2n_1<|i|\leq N} p_i^n h\right)\left(\sum_{|i|\leq n_c} \delta_c^i h - 1\right).$$

Since $\sum_{|i|\leq n_c} \delta_c^i h = \int_{-\frac{1}{2c}}^{\frac{1}{2c}} \delta_c(\sigma)\, d\sigma = 1$, we obtain

$$\sum_{|i|\leq N}\left(p_i^{n+1} - p_i^n\right) = 0,$$

and the result follows. ■

It follows from these lemmata that if $0 < s \leq T_0$ and $p_h^{c,N,0} \in \mathbb{D}^N$, then $p_h^{c,N,n} \in \mathbb{D}^N$ for any $n \geq 1$. Thus, we can correctly define the discrete semigroup $S_{h,s}^{c,N} : \mathbb{D}^N \to \mathbb{D}^N$ by $S_{h,s}^{c,N}\left(n, p_h^{c,N,0}\right) = p_h^{c,N,n}$. This map is continuous with respect to the initial data $p_h^{c,N,0}$.

Further, we will obtain an estimate of the solution $p_h^{c,N,n}$ in the norm of the space $\bar{l}^{1N}$.

Lemma 6. *Assume that $0 < s < T_0$. Then there exist $C, \delta > 0$ (independent of N and s) such that*

$$\left\|p_h^{c,N,n}\right\|_{\bar{l}^{1N}} \leq C + e^{-\delta s n}\left\|p_{c,h}^{N,0}\right\|_{\bar{l}^{1N}}, \tag{48}$$

for all $n \geq 1$ and $p_{c,h}^{N,0} \in \mathbb{D}^N$.

Proof. We multiply (42) by the vector $v = (|i|)_{|i|\leq \mathbb{N}}$. Arguing as in the proof of Lemma 16 in [8] we obtain

$$\sum_{|i|\leq N} \frac{p_i^{n+1} - p_i^n}{s}|i| + \frac{1}{T_0}\sum_{|i|\leq N} p_i^n |i| \leq \frac{D_h^N\left(p_h^{c,N,n}\right)}{\alpha}\sum_{i=-n_{h,c}}^{n_{h,c}} \delta_c^i |i| + \frac{1}{T_0}\sum_{|i|\leq 2n_1} p_i^n |i| + \left(D_h^N\left(p_h^{c,N,n}\right) + \frac{1}{c}\right)\frac{2p_0^n}{h^2}$$
$$\leq \frac{3}{2T_0 h^2} + \left(\frac{\alpha}{T_0} + \frac{1}{c}\right)\frac{2}{h^3} + \frac{1}{T_0 h^2} = K.$$

Thus,

$$\sum_{|i|\leq N} p_i^{n+1}|i| \leq \left(1 - \frac{s}{T_0}\right)\sum_{|i|\leq N} p_i^n |i| + sK, \quad \forall n \geq 0.$$

Since $0 < 1 - \frac{s}{T_0} < 1$, by induction we have

$$\sum_{|i|\leq N} p_i^{n+1} |i| \leq \left(1 - \frac{s}{T_0}\right)^{n+1} \sum_{|i|\leq N} p_i^0 |i| + sK \sum_{j=0}^{n} \left(1 - \frac{s}{T_0}\right)^j$$
$$\leq e^{\frac{-s}{T_0}(n+1)} \sum_{|i|\leq N} p_i^0 |i| + KT_0, \quad \forall n \geq 0.$$

Then using $\sum_{|i|\leq N} p_i \leq \frac{1}{h}$ we have

$$\left\| p_h^{c,N,n+1} \right\|_{\tilde{l}^{1N}} \leq C + e^{\frac{-s}{T_0}(n+1)} \left\| p_{c,h}^{N,0} \right\|_{\tilde{l}^{1N}}. \quad \blacksquare$$

Now we are ready to prove the main result of this section stating the convergence of the solutions of problem (42) to the solutions of problem (16).

For $s > 0$ denote by $p_h^{c,N,s} = \{p_h^{c,N,n}\}_{n\geq 0}$ the unique solution to problem (42) with initial data $p_{c,h}^{N,0}$.

Lemma 7. *Assume that $0 < s < T_0$. Let $T > 0$ and n_0 be such that $sn_0 \leq T < (n_0 + 1)s$. For any solution $p_h^{c,N,s}$ to (42) with initial data $p_h^{c,N,0} \in \mathbb{D}^N$ and any solution $p_h^{c,N}(\cdot)$ to (16) with initial data $\overline{p}_h^{c,N,0} \in \mathbb{D}^N$ it holds:*

$$\left\| p_h^{c,N}(t_{n+1}) - p_h^{c,N,n+1} \right\|_{\tilde{l}^{1N}} \leq e^{K_1(t_{n+1})} \left\| \overline{p}_h^{c,N,0} - p_h^{c,N,0} \right\|_{\tilde{l}^{1N}} + \left(e^{K_1(t_{n+1})} - 1\right) sD_1, \tag{49}$$

for $n = 0, \ldots, n_0 - 1$, where the constants K_1, D_1 can depend on $p_h^{c,N}(\cdot)$, $\left\| p_h^{c,N,0} \right\|_{\tilde{l}^{1N}}$, c, h, N and T, but not on s.

Proof. First we note by Lemma 13 in [8] $p_h^{c,N}(\cdot) \in C^2\left([0, T], \mathbb{R}^{2N+1}\right)$ and then

$$p_{h,i}^{c,N}(t_{n+1}) = p_{h,i}^{c,N}(t_n) + s\frac{dp_{h,i}^{c,N}(t_n)}{dt} + \frac{s^2}{2}\frac{d^2 p_{h,i}^{c,N}(\xi_i)}{dt^2},$$

for some $\xi_i \in]t_n, t_{n+1}[$. Define the map $G_h^{c,N} : \mathbb{R}^{2N+1} \times \mathbb{R}^{2N+1} \to \mathbb{R}^{2N+1}$ by

$$\left(G_h^{c,N}(p, q)\right)_i = -\left(D_h^N(p) + \frac{1}{c}\right)\left(A_h^N q\right)_i - \frac{1}{T_0}\chi_{\mathbb{Z}\setminus[-2n_1,2n_1]}(i)\, p_i + \frac{D_h^N(p)}{\alpha}\delta_c^i.$$

Since $\frac{d^2 p_h^{c,N}}{dt^2}$ is continuous, it is uniformly bounded in $[0, T]$. Then

$$\left\| p_h^{c,N}(t_{n+1}) - p_h^{c,N,n+1} \right\|_{\tilde{l}^{1N}}$$
$$\leq \left\| p_h^{c,N}(t_n) - p_h^{c,N,n} \right\|_{\tilde{l}^{1N}} + s\left\| G_h^N\left(p_h^{c,N}(t_n), p_h^{c,N}(t_n)\right) - G_h^N\left(p_h^{c,N,n}, p_h^{c,N,n+1}\right) \right\|_{\tilde{l}^{1N}} + s^2 M(T). \tag{50}$$

Following a similar argument as in [6, p. 2698] we obtain the estimate:

$$\left\| G_h^N(p_1, q_1) - G_h^N(p_2, q_2) \right\|_{\tilde{l}^{1N}} \leq K\left(\|q_1 - q_2\|_{\tilde{l}^{1N}} + \|p_1 - p_2\|_{\tilde{l}^{1N}}\left(\|q_2\|_{\tilde{l}^{1N}} + 1\right)\right), \tag{51}$$

for any $p_i, q_i \in \mathbb{D}^N$, where K depends on α, h, T_0 and c.

On the other hand, in view of Lemma 6 we get

$$\left\| p_h^{c,N,n} \right\|_{\tilde{l}^{1N}} \leq C + e^{-\delta sn} \left\| p_h^{c,N,0} \right\|_{\tilde{l}^{1N}} \leq \overline{C}, \quad \text{for all } n \geq 0, \tag{52}$$

where $\overline{C}$ does not depend on s.

Therefore, using (50)–(52) we have

$$\left\| p_h^{c,N}(t_{n+1}) - p_h^{c,N,n+1} \right\|_{\tilde{l}^{1N}} \leq \left\| p_h^{c,N}(t_n) - p_h^{c,N,n} \right\|_{\tilde{l}^{1N}}$$
$$+ sK\left(\left\| p_h^{c,N}(t_n) - p_h^{c,N,n+1} \right\|_{\tilde{l}^{1N}} + (\overline{C} + 1)\left\| p_h^{c,N}(t_n) - p_h^{c,N,n} \right\|_{\tilde{l}^{1N}}\right) + s^2 M(T)$$
$$\leq \left\| p_h^{c,N}(t_n) - p_h^{c,N,n} \right\|_{\tilde{l}^{1N}}$$
$$+ sK_1\left(\left\| p_h^{c,N}(t_n) - p_h^{c,N,n} \right\|_{\tilde{l}^{1N}} + \left\| p_h^{c,N,n+1} - p_h^{c,N,n} \right\|_{\tilde{l}^{1N}}\right) + s^2 M(T).$$

Now by (42), (52) and $\left|D_h^N\left(p_h^{c,N,n}\right)\right| \leq \frac{\alpha}{T_0}$ we deduce that

$$\left\|p_h^{c,N,n+1} - p_h^{c,N,n}\right\|_{\bar{l}^{1N}} \leq sK_2, \tag{53}$$

so that

$$\left\|p_h^{c,N}(t_{n+1}) - p_h^{c,N,n+1}\right\|_{\bar{l}^{1N}} \leq (1+sK_1)\left\|p_h^{c,N}(t_n) - p_h^{c,N,n}\right\|_{\bar{l}^{1N}} + s^2K_3.$$

By induction we obtain

$$\begin{aligned}\left\|p_h^{c,N}(t_{n+1}) - p_h^{c,N,n+1}\right\|_{\bar{l}^{1N}} &\leq (1+sK_1)^{n+1}\left\|p_h^{c,N}(0) - p_h^{c,N,0}\right\|_{\bar{l}^{1N}} + \frac{(1+sK_1)^{n+1}-1}{sK_1}s^2K_3\\ &\leq e^{K_1(n+1)s}\left\|p_h^{c,N}(0) - p_h^{c,N,0}\right\|_{\bar{l}^{1N}} + \left(e^{K_1(n+1)s}-1\right)\frac{sK_3}{K_1},\end{aligned}$$

for $n = 0, \ldots, n_0 - 1$. ■

We consider the initial data $p_{c,h}^{N,0}$ given by (17). Then we define the step functions

$$p_h^{c,N,s}(t,\sigma) = \sum_{n\in\mathbb{Z}^+}\sum_{i\in\mathbb{Z}} p_{h,i}^{c,N,n}\chi_{[t_n,t_{n+1})}(t)\chi_{I_i}(\sigma),$$

where $p_h^{c,N,s} = \left\{p_{h,i}^{c,N,n}\right\}_{|i|\leq N,\ n\geq 0}$ is the unique solution to problem (42) with initial data $p_{c,h}^{N,0}$ and $p_{h,i}^{c,N,n} = 0$ if $|i| > N$.

Lemma 8. *Let $T > 0$. Then $p_h^{c,N,s}$ converges to $p_h^{c,N}$ in $C([0,T],L^2(\mathbb{R}))$ as $s \to 0^+$, where $p_h^{c,N}(t,\sigma)$ is the function defined in (18) with the same initial data $p_{c,h}^{N,0}$.*

Proof. Let $\overline{T} = T + 1$ and $t \in [0,T]$. For any $0 < s < 1$ there exists $n(s)$ such that $t \in [t_{n(s)}, t_{n(s)+1}) \subset [0,\overline{T}]$. Then

$$\begin{aligned}\left\|p_h^{c,N,s}(t,\cdot) - p_h^{c,N}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &= h\sum_{i\in\mathbb{R}}\left|p_{h,i}^{c,N,n(s)} - p_{h,i}^{c,N}(t)\right|^2\\ &\leq 2h\sum_{i\in\mathbb{R}}\left|p_{h,i}^{c,N,n(s)} - p_{h,i}^{c,N}\left(t_{n(s)}\right)\right|^2 + 2h\sum_{i\in\mathbb{R}}\left|p_{h,i}^{c,N}\left(t_{n(s)}\right) - p_{h,i}^{c,N}(t)\right|^2.\end{aligned}$$

In view of $t_{n(s)} \to t$ and $p_h^{c,N} \in C([0,T],L^2(\mathbb{R}))$ there exists $s_1 > 0$, which does not depend on $t \in [0,T]$, such that $\sum_{i\in\mathbb{R}}\left|p_{h,i}^{c,N}\left(t_{n(s)}\right) - p_{h,i}^{c,N}(t)\right|^2 < \frac{\varepsilon}{4h}$ if $s < s_1$. Applying Lemma 7 we obtain the existence of $s_2 > 0$ (not depending on $t \in [0,T]$) such that

$$\begin{aligned}\sum_{i\in\mathbb{R}}\left|p_{h,i}^{c,N,n(s)} - p_{h,i}^{c,N}\left(t_{n(s)}\right)\right|^2 &= \left\|p_h^{c,N,n(s)} - p_h^{c,N}\left(t_{n(s)}\right)\right\|^2_{\mathbb{R}^{2N+1}}\\ &\leq C_1\left\|p_h^{c,N,n(s)} - p_h^{c,N}\left(t_{n(s)}\right)\right\|^2_{\bar{l}^{1N}}\\ &\leq C_2\left(e^{K_1\overline{T}}-1\right)s \leq \frac{\varepsilon}{4h},\end{aligned}$$

if $s < s_2$. Thus, there is $s_3 > 0$ such that for any $t \in [0,T]$ and any $s < s_3$ one has

$$\left\|p_h^{c,N,s}(t,\cdot) - p_h^{c,N}(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq \varepsilon. \quad ■$$

Finally, we will put together the three convergences from Section 2 and this new one.

We take sequences $c_m \to \infty$, $c_m \in \mathbb{N}$, $h_{n,m} \to 0$ (as $n \to \infty$), such that $\frac{1}{2c_m h_n} = n_{h_n,c_m} \in \mathbb{N}$. Let us consider the sequence of initial data $p^0, p_c^0, p^0_{c_m,h_{n,m}}$ and $p^{N,0}_{c_m,h_{n,m}}$ described in Section 2. Let $p, p^{c_m}, p^{c_m}_{h_{n,m}}, p^{c_m,N}_{h_{n,m}}, p^{c_m,N,s}_{h_{n,m}}$ be the solutions to problems (1), (4), (8), (16) and (42), respectively, with the corresponding initial conditions $p^0, p^0_{c_m}, p^0_{c_m,h_{n,m}}$ and $p^{N,0}_{c_m,h_{n,m}}$.

Then from the results in Section 2 and Lemma 8 we have the following convergences:

$$\begin{aligned}&p^{c_m,N,s}_{h_{n,m}} \to p^{c_m,N}_{h_{n,m}} \quad \text{in } C([0,T],L^2(\mathbb{R})) \text{ as } s \to 0^+,\\ &p^{c_m,N}_{h_{n,m}} \to p^{c_m}_{h_{n,m}} \quad \text{in } C([0,T],L^2(\mathbb{R})) \text{ as } N \to \infty,\\ &p^{c_m}_{h_{n,m}} \to p^{c_m} \quad \text{in } C([0,T],L^2(\mathbb{R})) \text{ as } n \to \infty,\end{aligned}$$

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Table 1
$\alpha = 0.6;\ h = 0.001;\ N = 10\,000;\ s = 0.1.$

Value of c	Time	$\Vert p_h^{c,N,n} \Vert_{L^1(\mathbb{R})}$	$\Vert p_h^{c,N,n} - \bar{p}_h^{c,N} \Vert_{L^2(\mathbb{R})}$	$\Vert \bar{p}_h^{c,N} - \bar{p} \Vert_{L^2(\mathbb{R})}$
$c = 50$	$t = 0.0$	1.0	1.08085618491	0.125390802601
	$t = 0.1$	1.0	0.882963512592	
	$t = 0.5$	1.0	0.494129450788	
	$t = 1.2$	1.0	0.23640717103	
	$t = 4.8$	1.0	0.0192086602722	
	$t = 10.0$	1.0	0.001139880963	
$c = 125$	$t = 0.0$	1.0	1.10354101408	0.0831586822063
	$t = 0.1$	1.0	0.907744048993	
	$t = 0.6$	1.0	0.46468164988	
	$t = 1.3$	1.0	0.241399232916	
	$t = 6.2$	1.0	0.019267902446	
	$t = 10.0$	1.0	0.00432273450475	
$c = 500$	$t = 0.0$	1.0	1.12440163552	0.0400867762899
	$t = 0.1$	1.0	0.930302934472	
	$t = 0.6$	1.0	0.490055110457	
	$t = 1.5$	1.0	0.234584869029	
	$t = 9.2$	1.0	0.0196430452113	
	$t = 10.0$	1.0	0.0163811084573	

$$p^{c_m} \to p \quad \text{in } C([0,T], L^2(\mathbb{R}) \cap \bar{L}^1(\mathbb{R})) \text{ as } m \to \infty.$$

Hence, we obtain the iterate limit

$$\lim_{m\to\infty} \lim_{n\to\infty} \lim_{N\to\infty} \lim_{s\to 0^+} p_{h_{n,m}}^{c_m,N,s} = p \quad \text{in } C([0,T], L^2(\mathbb{R})).$$

5. Numerical simulations

This section is devoted to illustrate graphically the numerical simulations given by the discrete finite-difference approximations from the previous section. We shall consider two different initial conditions, the first one being a Gaussian density and the second one being a constant density inside an interval and zero outside of it. Let us point out that the examples reported below are only a small sample intended for showing the behaviour of the approximations, and that similar results are obtained when initial conditions are changed.

Gaussian density

Consider the problem (1) with initial condition given by the Gaussian function having a mean of −2 and a standard deviation of 0.4:

$$p^0(\sigma) = \frac{1}{0.4\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\sigma+2}{0.4}\right)^2}.$$

Let us fixed the parameters $\alpha = 0.6$, $h = 0.001$, $s = 0.1$ and $N = 10^4$. Figs. 1–3 depict the solution $p_h^{c,N,n}$ of (42) (with initial data $p_{c,h}^{N,0}$ given by (17)) for different values of n and c and the fixed points $\bar{p}, \bar{p}_h^{c,N}$ of problems (1) and (42), respectively. Table 1 collects, on the one hand, the numeric measures describing how fast the solutions of the discrete system (42) converge to the unique fixed point of the system and, on the other hand, the distance in the space $L^2(\mathbb{R})$ of the unique fixed point $\bar{p}_h^{c,N}$ of system (42) to the unique fixed point $\bar{p}$ of Eq. (1) with positive value of $D(\bar{p})$.

Uniform density on an interval

Consider the problem (1) with initial condition given by the function

$$p^0(\sigma) = \begin{cases} 1, & \text{if } \sigma \in [-2.5, -1.5], \\ 0, & \text{otherwise.} \end{cases}$$

Let us fixed the same parameters $\alpha = 0.6$, $h = 0.001$, $s = 0.1$ and $N = 10^4$. Figs. 4–6 and Table 2 show similar results as in the previous example.

These numerical simulations suggest that the fixed point $\bar{p}$ is not only asymptotically stable but every solution with initial data p_0 satisfying $D(p_0) > 0$ converges to it as time tends to $+\infty$.

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Table 2
$\alpha = 0.6$; $h = 0.001$; $N = 10\,000$; $s = 0.1$.

Value of c	Time	$\|p_h^{c,N,n}\|_{L^1(\mathbb{R})}$	$\|p_h^{c,N,n} - \bar{p}_h^{c,N}\|_{L^2(\mathbb{R})}$	$\|\bar{p}_h^{c,N} - \bar{p}\|_{L^2(\mathbb{R})}$
$c = 50$	$t = 0.0$	1.0	1.20499712827	0.125390802601
	$t = 0.1$	1.0	0.931089684917	
	$t = 0.6$	1.0	0.447116512478	
	$t = 1.2$	1.0	0.239084976463	
	$t = 4.8$	1.0	0.0192266615085	
	$t = 10.0$	1.0	0.00112649297488	
$c = 125$	$t = 0.0$	1.0	1.22968032642	0.0831586822063
	$t = 0.1$	1.0	0.954905314626	
	$t = 0.6$	1.0	0.473224445693	
	$t = 1.3$	1.0	0.243729229683	
	$t = 6.2$	1.0	0.0192858187493	
	$t = 10.0$	1.0	0.00431010522074	
$c = 500$	$t = 0.0$	1.0	1.24998608633	0.0400867762899
	$t = 0.1$	1.0	0.976049250289	
	$t = 0.6$	1.0	0.498354753926	
	$t = 1.5$	1.0	0.236303475973	
	$t = 9.2$	1.0	0.0196628831918	
	$t = 10.0$	1.0	0.0163955298278	

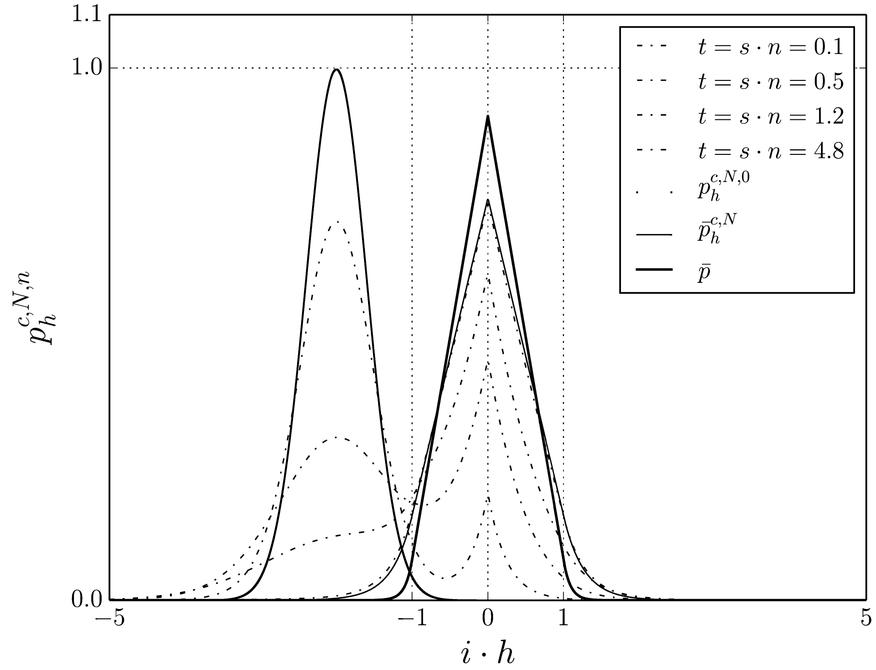

Fig. 1. $\alpha = 0.6$; $c = 50.0$; $h = 0.001$; $N = 10\,000$; $s = 0.1$.

6. Conclusions

In this paper we have completed the mathematical study of the numerical scheme which was introduced in [8] in order to study the non-Newtonian suspension model (1). In that paper, as explained in Section 2, the convergence of the solutions of a sequence of approximative problems to the solution of the initial-value problem (1)–(2) was proved. In the present paper, we have extended such results in two ways.

First, we have checked that in the approximative equations there exists a unique fixed point, and that it converges to the unique equilibrium of the original equation with support outside the interval $[-1, 1]$.

Second, we have considered a full discretized system (for both spatial and temporal variables), implementing in this way numerical simulations of the solutions. What is more, these simulations allow for formulating an interesting and realistic hypothesis about the asymptotic behaviour of solutions of the original equations.

The main advantage of our approximative scheme is the fact that we have been able to perform a rigorous mathematical analysis of the convergence of the solutions and, moreover, we have shown that the essential features of the dynamics of the approximative systems are the same as in the original equation (in particular, the limit point of the approximative solutions

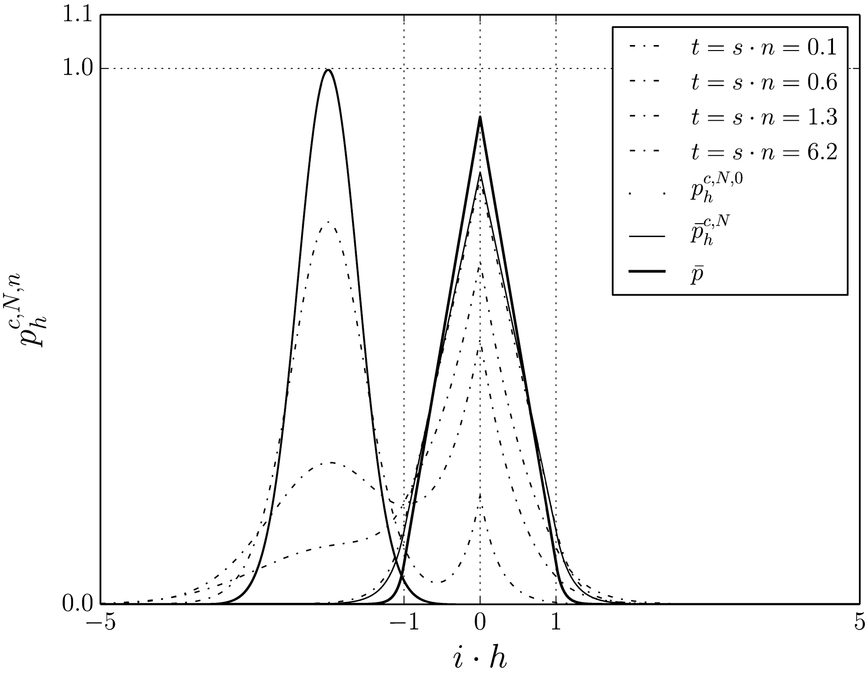

Fig. 2. $\alpha = 0.6$; $c = 125.0$; $h = 0.001$; $N = 10\,000$; $s = 0.1$.

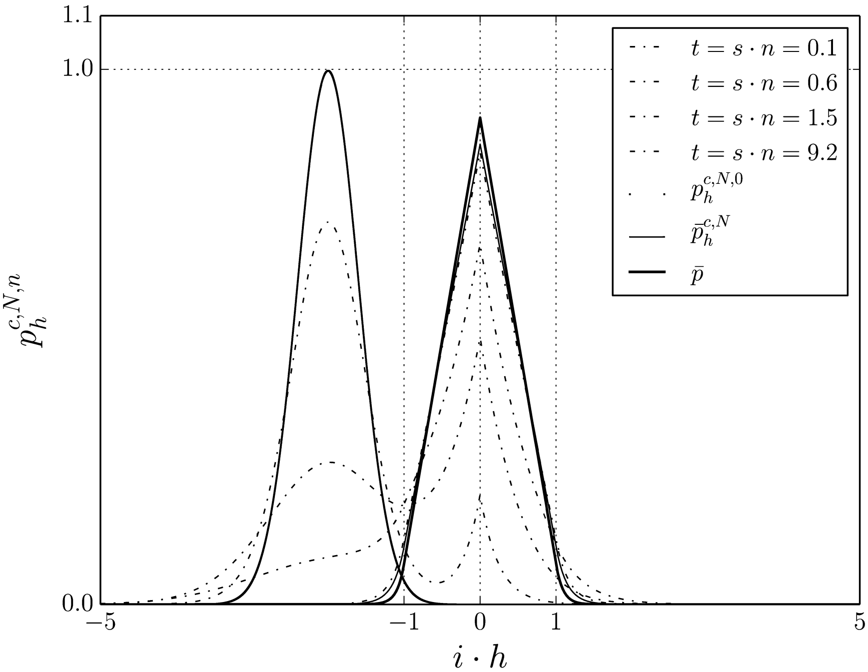

Fig. 3. Parameters: $\alpha = 0.6$; $c = 500.0$; $h = 0.001$; $N = 10\,000$; $s = 0.1$.

as time goes to infinity is close to the unique equilibrium of the original equation with support outside the interval $[-1, 1]$). Such results are much more difficult or even impossible to achieve for more complex approximations. Of course, since the approximations are simpler, the convergence is slower as well, which is a drawback. However, as the dynamics in our model is quite simple (it seems that all the solutions with support outside the interval $[-1, 1]$ converge to a given fixed point as time increases), this is not relevant.

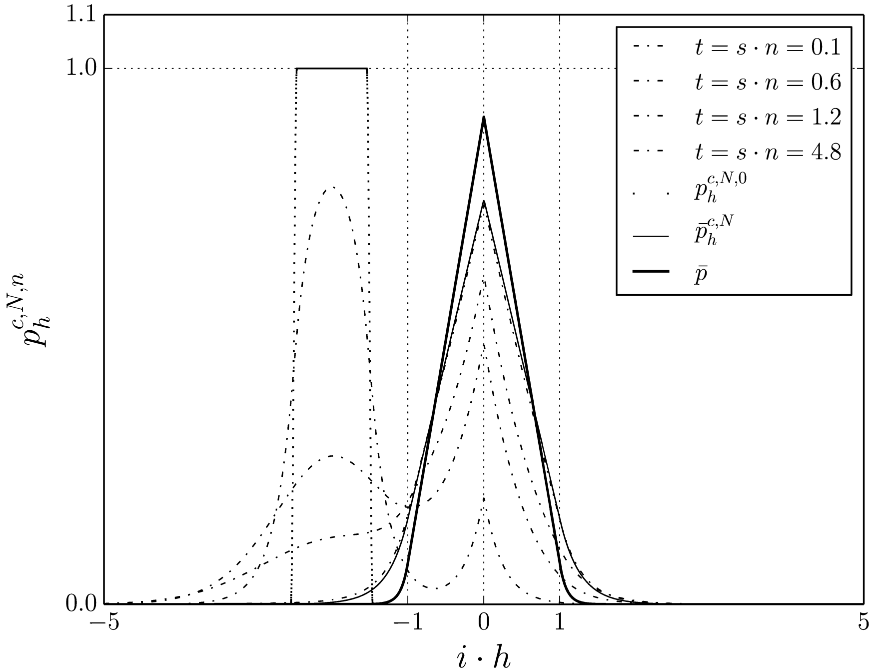

Fig. 4. $\alpha = 0.6$; $c = 50.0$; $h = 0.001$; $N = 10\,000$; $s = 0.1$.

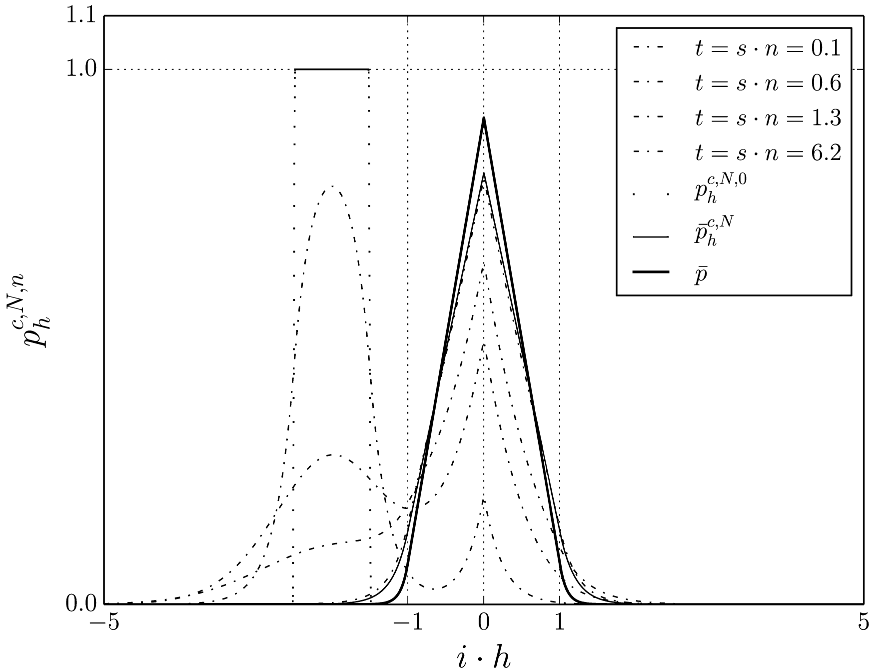

Fig. 5. $\alpha = 0.6$; $c = 125.0$; $h = 0.001$; $N = 10\,000$; $s = 0.1$.

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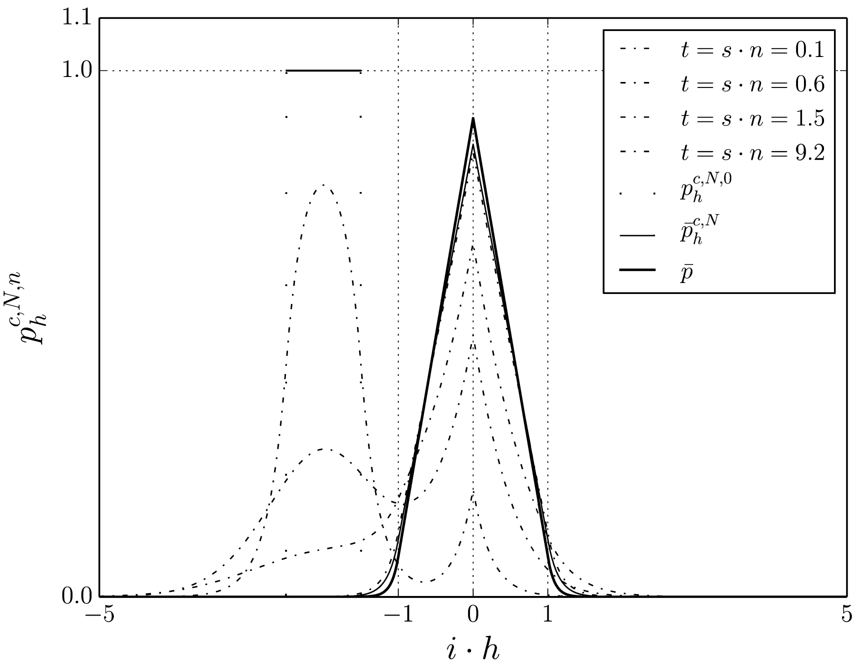

Fig. 6. $\alpha = 0.6$; $c = 500.0$; $h = 0.001$; $N = 10\,000$; $s = 0.1$.

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