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Construction of the fundamental solution of disturbed parabolic equation

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Abstract

In present paper the parabolic equation solution is built. The construction is reduced to iterative procedure. And convergence of the latter is proven.

Résumé

Dans un travail presenté on a construit la solution d'equation parabolique. Cette construction est reduite a procédé d'iteration. On demonstre la convergence de le dernier.

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1. Introduction

In presented work the problem of construction of fundamental solution $e^{t(L+L_1)}$ of equation

$$\frac{du}{dt} = (L + L_1)u$$

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is considered, where L and L_1 are elliptic operators, and the properties of evolution operator e^{tL} of a non-disturbed equation

$$\frac{du}{dt} = Lu$$

are assumed to be known.

If the coefficients L and L_1 are constant, then

$$e^{t(L+L_1)} = e^{tL}e^{tL_1}. \tag{1}$$

For the non-constant coefficients, satisfying some set of restrictions, we will prove that the right part of (1) under small t is quite good approximation for the evolution operator in the left part, that is,

$$e^{t(L+L_1)} = e^{tL}e^{tL_1} + A(t), \quad A(0) = 0,$$

and for the construction of family of operators $A(t)$ the iteration procedure will be proposed. In this work two examples of parabolic equations are considered:

1. L is the elliptic operator with constant coefficients,

$$Lu = \frac{1}{2}a^{jk}\frac{\partial^2 u}{\partial x^j \partial x^k},$$

the perturbation L_1 has a form

$$L_1 u = \frac{1}{2}b^{jk}(x)\frac{\partial^2 u}{\partial x^j \partial x^k}.$$

2. $L = \frac{1}{2}\Delta$, where Δ is Laplace–Beltrami operator on complete simply connected Riemann manifold M of non-positive curvature with dimension n, and metrics tensor $g_{jk}(x)$, distance ρ and volume σ, that is

$$Lu(x) = \frac{1}{2}\operatorname{div}\operatorname{grad} u(x),$$

and perturbation has a form

$$L_1 u(x) = \frac{1}{2}\operatorname{div} b(x)\operatorname{grad} u(x).$$

In the both examples

$$\left(e^{tL}f\right)(x) = \int f(y)p_0(t,x,y)\,dy,$$

where p_0 is a fundamental solution of non-disturbed equation. The aim of presented paper is to propose and to ground well a constructing of the function p, obtained from the equality

$$\left(e^{t(L+L_1)}f\right)(x) = \int f(y)p(t,x,y)\,dy.$$

Above-mentioned function $p(t,x,y)$ will be searched in the form

$$p(t,x,y) = m(t,x,y) + \int_0^t d\tau \int m(t-\tau,x,z)r(\tau,z,y)\,dz, \tag{2}$$

where the initial approximation

$$m(t,x,y)=\int p(t,z,y)p_1(t,x,z)\,\sigma(dz), \tag{3}$$

the function p_1 is an approximation of the kernel of integral operator e^{tL_1}, and $r(t,x,y)$ is a function being subject to be obtained.

A procedure of constructing of fundamental solution is analogous to the same procedure in parametrix method, namely: Eq. (2) is reduced to Volterra's integral equation for the function r

$$r(t,x,y)=M(t,x,y)+\int_0^t d\tau\int M(t-\tau,x,z)r(\tau,z,y)\,dz, \tag{4}$$

where an error $M(t,x,y)=(L+L_1)m-\frac{\partial m}{\partial t}$, and solution r of Eq. (3) has a structure $r(t,x,y)=\sum_{n=0}^{\infty}r_n(t,x,y)$, and each iteration is calculated with respect to recurrent formula:

$$r_0(t,x,y)=M(t,x,y),$$

$$r_{n+1}(t,x,y)=\int_0^t d\tau\int M(t-\tau,x,z)r_n(\tau,z,y)\,dz.$$

Convergence of the last integral and the series $\sum r_n$ is determined by properties of the error $M(t,x,y)$: it must have an integrable with respect to t singularity. Above-mentioned property holds under some restrictions on coefficients of the operators L and L_1.

Since the initial approximation $m(t,x,y)$ (and hence, the error $M(t,x,y)$ as well) is defined as an integral, for transforming of the error and its estimating the integration by the parts will be applied:

$$\int \operatorname{div} V(z)\,\mu(dz)=-\int\bigl(\Lambda(z),V(z)\bigr)\,\mu(dz),$$

where the role of the measure μ the relation $p_1(t,x,z)\sigma(dz)$ has:

$$\int \operatorname{div} V(z)p_1(t,x,z)\,\sigma(dz)=-\int\bigl(\Lambda(z),V(z)\bigr)p_1(t,x,z)\,\sigma(dz). \tag{5}$$

Here a logarithmic derivative is $\Lambda(t,x,z)=\operatorname{grad}_\varepsilon \ln p_1(t,x,z)$.

1.1. Perturbation of the constant operator

Let's demonstrate the described method in particular case, considering a parabolic equation

$$\frac{\partial u}{\partial t}=\frac{1}{2}\operatorname{tr}\bigl(A+B(x)\bigr)u'',\quad x\in R^n, \tag{6}$$

where A is a constant operator in R^n, and a positive operator $B(x)$ satisfies the conditions:

(1) a positive operator K_0 exists, such that

$$K(x) = A^{-1/2}B(x)A^{-1/2} \leqslant K_0 \leqslant \delta I, \quad \delta < 1;$$

(2) $\|B(x)B^{-1}(y)\| < \text{Const}$;

(3) the fist two derivatives of the operator $B(x)$ are bounded:

$$\|B'(x)h\| \leqslant c_1\|h\|,$$
$$\|B''(x)kh\| \leqslant c_2\|k\| \cdot \|h\|.$$

From condition 2 the boundedness of the ratio $\frac{\det B(x)}{\det B(y)}$ follows.

The solution of non-disturbed equation is

$$p_0(t,x,y) = \frac{1}{(2\pi t)^{n/2}\sqrt{\det A}} \exp\left\{-\frac{(A^{-1}(x-y), x-y)}{2t}\right\}.$$

Put

$$p_1(t,x,y) = \frac{1}{(2\pi t)^{n/2}\sqrt{\det B(y)}} \exp\left\{-\frac{(B^{-1}(x)(x-y), x-y)}{2t}\right\},$$

then zero approximation is

$$m(t,x,y) = \int\limits_{R^n} p(t,z,y)\frac{1}{(2\pi t)^{n/2}\sqrt{\det B(z)}} \exp\left\{-\frac{(B^{-1}(x)(x-z), x-z)}{2t}\right\} dz.$$

From conditions 1 and 2 an estimate follows:

$$\begin{aligned} m(t,x,y) &< \frac{c}{\sqrt{\det(I+K(x))}} \\ &\quad \times \exp\left\{\frac{1}{2t}\left(A^{-1/2}K(x)A^{-1/2}(x-y), x-y\right)\right\} p_0(t,x,y) \\ &< c\exp\left\{\frac{1}{2t}\left(A^{-1/2}K_0A^{-1/2}(x-y), x-y\right)\right\} p_0(t,x,y). \end{aligned}$$

Error

$$M(t,x,y) = \frac{1}{2}\operatorname{tr}\bigl(A + B(x)\bigr)m''_{xx} - \frac{\partial m}{\partial t} = I_1 + I_2,$$

where

$$I_1 = \int\limits_{R^n}\left(\frac{1}{2}\operatorname{tr} B(x)p_1''(t,x,z) - \frac{\partial p_1}{\partial t}(t,x,z)\right)p_0(t,z,y)\,dz,$$

$$I_2 = \int\limits_{R^n}\left(\frac{1}{2}\operatorname{tr} Ap_1''(t,x,z)\cdot p(t,z,y) - \frac{\partial p_0}{\partial t}(t,z,y)p_1(t,x,z)\right)dz,$$

and differentiation is done with respect to variable x. For the next transformations let's reduce I_2 to the form

$$I_2 = \frac{1}{2}\int\limits_{R^n}\bigl(\operatorname{tr} Ap_1''(t,x,z)p_0(t,z,y) - \operatorname{tr} Ap''_{0xx}(t,z,y)p_1(t,x,z)\bigr)\,dz,$$

and integrate the second item twice by the parts, see (5):

$$
\begin{aligned}
&-\frac{1}{2}\int\limits_{R^n} \operatorname{tr}\bigl(Ap'_{0x}(t,z,y)\bigr)'_x p_1(t,x,z)\,dz \\
&= -\frac{1}{2}\int\limits_{R^n} \operatorname{div}_x\bigl(Ap'_{0x}(t,z,y)\bigr)p_1(t,x,z)\,dz \\
&= \frac{1}{2}\int\limits_{R^n} \bigl(p'_{0x}(t,z,y), A\Lambda(t,x,z)\bigr)p_1(t,x,z)\,dz \\
&= -\frac{1}{2}\int\limits_{R^n} \bigl[\bigl(A\Lambda(t,x,z),\Lambda(t,x,z)\bigr) + \operatorname{div}_z A\Lambda(t,x,z)\bigr]p_0(t,z,y)p_1(t,x,z)\,dz.
\end{aligned}
$$

Lemma 1. *A logarithmic derivative $\Lambda(t,x,z)$ of the measure $p_1(t,x,z)\,dz$ is defined by the equality*:

$$
\bigl(\Lambda(t,x,z),h\bigr) = -\frac{1}{2}\operatorname{tr} B'(z)hB^{-1}(z) + \frac{1}{t}\bigl(B^{-1}(x)(x-z),h\bigr).
$$

Proof. From definition

$$
\bigl(\Lambda(t,x,z),h\bigr) = -\frac{1}{2}d_h \ln\det B(z) + \frac{1}{t}\bigl(B^{-1}(x)(x-z),h\bigr),
$$

where d_h is a differential of function along vector h with respect to variable z.

Differentiating of the equality implies

$$
\ln\det B(z) = \operatorname{tr}\int\limits_0^1 \bigl(B(z)-I\bigr)\bigl(I+\tau(B(z)-I)\bigr)^{-1}\,d\tau,
$$

and we get

$$
d_h \ln\det B(z) = \operatorname{tr} B'(z)h\bigl(B(z)-I\bigr)^{-1}\int\limits_0^1 \bigl(B(z)-I\bigr)\bigl(I+\tau(B(z)-I)\bigr)^{-2}\,d\tau,
$$

thus, the statement of the lemma follows from the fact that the relation under integral sign equals

$$
-\frac{d}{d\tau}\bigl(I+\tau(B(z)-I)\bigr)^{-1}. \qquad \square
$$

Corollary.

$$
\begin{aligned}
I_2 = \frac{1}{2}\int\limits_{R^n} &\left[\frac{\operatorname{tr} Ap''_1(t,x,z)}{p_1(t,x,z)} - \bigl(A\Lambda(t,x,z),\Lambda(t,x,z)\bigr) - \operatorname{div}_x A\Lambda(t,x,z)\right] \\
&\times p_0(t,z,y)p_1(t,x,z)\,dz.
\end{aligned}
$$

Lemma 2. *Error $M(t,x,y)$ satisfies the following estimate*:

$$\left|M(t,x,y)\right| \leqslant \frac{c}{\sqrt{t}} \int_{R^n} \left(1+\frac{\|x-z\|^4}{t^2}\right) p_0(t,z,y) p_1(t,x,z)\,dz.$$

Proof. Differentiating p_1, we reduce I_1 to the form ($\{e_k\}$ is an orthogonal basis in R^n):

$$\begin{aligned} I_1 = \frac{1}{2}\int_{R^n}\sum_k \bigg[& \frac{1}{4t^2}\big((B^{-1})' e_k(x-z), x-z\big)\big((B^{-1})' B e_k(x-z), x-z\big) \\ & + \frac{1}{2t^2}\big((B^{-1})' e_k(x-z), x-z\big)(x-z, e_k) \\ & + \frac{1}{2t^2}\big((B^{-1})' B e_k(x-z), x-z\big)\big(B^{-1}(x-z), e_k\big) \\ & - \frac{1}{2t}\big((B^{-1})'' e_k B e_k(x-z), x-z\big) - \frac{1}{t}\big((B^{-1})' e_k(x-z), B e_k\big) \\ & - \frac{1}{t}\big((B^{-1})' B e_k(x-z), e_k\big)\bigg] p_0(t,z,y) p_1(t,x,z)\,dz. \end{aligned}$$

From lemma's conditions the relation in brackets satisfies the estimate:

$$c\left(\frac{\|x-z\|^4+\|x-z\|^3}{t^2}+\frac{\|x-z\|}{t}\right) < \frac{c}{\sqrt{t}}\left(1+\frac{\|x-z\|^4}{t^2}\right),$$

that is,

$$I_1 < \frac{c}{\sqrt{t}} \int_{R^n} \left(1+\frac{\|x-z\|^4}{t^2}\right) p_0(t,z,y) p_1(t,x,z)\,dz.$$

For estimating I_2 (see corollary to Lemma 1) we note that

$$\begin{aligned} \operatorname{div}_z A\Lambda(t,x,z) &= \operatorname{tr} A\Lambda'_z(t,x,z) \\ &= \operatorname{tr} A\bigg[-\frac{1}{2}\operatorname{tr} B''(z)(\cdot)B^{-1}(z) \\ &\qquad + \frac{1}{2}\operatorname{tr} B'(z)(\cdot)B^{-1}(z)B'(z)(\cdot)B^{-1}(z) - \frac{1}{t}B^{-1}(x)\bigg] \\ &= -\frac{1}{t}\operatorname{tr} AB^{-1}(x) + \phi(z), \end{aligned}$$

where the item ϕ is bounded in lemma's statement. The relation under integral sign in $I_2(A\Lambda(t,x,z), \Lambda(t,x,z))$ allows a presentation:

$$\big(A\Lambda(t,x,z), \Lambda(t,x,z)\big) = \frac{1}{t^2}\big(B^{-1}(x)AB^{-1}(x)(x-z), x-z\big) + \psi(t,x,z),$$

where

$$\left|\psi(t,x,z)\right| < \frac{c}{\sqrt{t}} \int_{R^n} \left(1+\frac{|x-z|}{\sqrt{t}}\right).$$

Calculating $Ap_1''(t,x,z)$, we see that under integral sign in I_2 the relation satisfying desired estimate, remains. □

Lemma 3. *For error $M(t,x,y)$ the following estimate holds*:

$$\left|M(t,x,y)\right| < \frac{c}{\sqrt{t}} \exp\left\{\frac{1}{2t}\left(A^{-1/2}K_0A^{-1/2}(x-y), x-y\right)\right\} p_0(t,x,y).$$

Proof. Let's use the equality

$$p_0(t,x,y) = \phi(t,z,x,y)p(t,x,y),$$

where

$$\begin{aligned}\phi(t,z,x,y) &= \exp\left\{\frac{1}{t}\left(A^{-1}(y-x), z-x\right)\right\} \exp\left\{-\frac{1}{2t}\left(A^{-1}(z-x), z-x\right)\right\}\\ &= \phi_1(t,x,y,z)\phi_2(t,x,z).\end{aligned}$$

From Lemma 2 the inequality follows:

$$\begin{aligned}\left|M(t,x,y)\right| &< \frac{c}{\sqrt{t}} p_0(t,x,y) \int\limits_{R^n} \left(1 + \frac{\|x-z\|^4}{t^2}\right)\\ &\quad \times \phi_2(t,x,z)\phi_1(t,x,y,z)p_1(t,x,z)\,dz,\end{aligned}$$

and from boundedness of the product $\frac{\|x-z\|^4}{t^2}\phi_2(t,x,z)$ the estimate for error gets the form

$$\left|M(t,x,y)\right| < \frac{c}{\sqrt{t}} p_0(t,x,y) \int\limits_{R^n} \exp\left\{\frac{1}{t}\left(A^{-1}(z-x), y-x\right)\right\} p_1(t,x,z)\,dz.$$

Substituting $z \to u$, $u = \frac{1}{\sqrt{t}}B^{-1/2}(x)(z-x)$, $z = x + \sqrt{t}\,B^{1/2}(x)u$, we get an inequality

$$\begin{aligned}\left|M(t,x,y)\right| &< \frac{c}{\sqrt{t}} p_0(t,x,y)\\ &\quad \times \int\limits_{R^n} \exp\left\{\left(u, \frac{1}{\sqrt{t}}B^{1/2}(x)A^{-1}(y-x)\right)\right\} \sqrt{\frac{\det B(x)}{\det B(z)}}\,\mu(du),\end{aligned}$$

where μ is a canonical Gaussian measure in R^n.

Calculation of the last integral (after estimating of the ratio of determinants) leads to an inequality

$$M(t,x,y) < \frac{c}{\sqrt{t}} p_0(t,x,y) \exp\left\{\frac{1}{2t}\left(A^{-1}B(x)A^{-1}(y-x), y-x\right)\right\},$$

and the statement of lemma follows. □

Theorem 1. *Fundamental solution $p(t,x,y)$ of disturbed equation* (6) *for $t \in (0,T]$ satisfies an inequality*

$$p(t,x,y) < c\exp\left\{\frac{(A^{-1/2}K_0A^{-1/2}(x-y), x-y)}{2t}\right\} p_0(t,x,y).$$

Proof. Let us construct $p_0(t,x,y)$ as a solution of integral equation (2). Estimating an iteration $r_1(t,x,y)$ of Eq. (4):

$$\begin{aligned}\left|r_1(t,x,y)\right| &< \int_0^t d\tau \int_{R^n} \left|M(\tau,z,y)M(t-\tau,x,z)\right| dz \\ &< c^2 \int_0^t \frac{d\tau}{\sqrt{\tau(t-\tau)}} \int_{R^n} \exp\left\{\frac{(A^{-1/2}K_0A^{-1/2}(z-y),z-y)}{2\tau}\right. \\ &\quad \left. + \frac{(A^{-1/2}K_0A^{-1/2}(z-x),z-x)}{2(t-\tau)}\right\} p_0(\tau,z,y)p_0(t-\tau,x,z)\,dz.\end{aligned}$$

We substitute

$$A^{-1/2}\left(z\sqrt{\frac{t}{\tau(t-\tau)}} - x\sqrt{\frac{\tau}{t(t-\tau)}} - y\sqrt{\frac{t-\tau}{t\tau}}\right) = u,$$

and hence relations

$$\begin{aligned}\frac{z-x}{\sqrt{t-\tau}} &= \sqrt{\frac{\tau}{t}}\,A^{1/2}u + \frac{\sqrt{t-\tau}}{t}(y-x), \\ \frac{z-y}{\sqrt{\tau}} &= \sqrt{\frac{t-\tau}{t}}\,A^{1/2}u + \frac{\sqrt{\tau}}{t}(x-y),\end{aligned} \tag{7}$$

transform a space integral in form

$$\begin{aligned}&p_0(t,x,y)\exp\left\{\frac{(A^{-1/2}K_0A^{-1/2}(z-y),z-y)}{2t}\right\} \int_{R^n} \exp\left\{\frac{(K_0u,u)}{2}\right\}\mu(du) \\ &\quad = \frac{1}{\sqrt{\det(I-K_0)}}\,p_0(t,x,y)\exp\left\{\frac{(A^{-1/2}K_0A^{-1/2}(z-y),z-y)}{2t}\right\},\end{aligned}$$

and from this

$$\left|r_1(t,x,y)\right| < c^2c_1\pi p_0(t,x,y)\exp\left\{\frac{(A^{-1/2}K_0A^{-1/2}(z-y),z-y)}{2t}\right\},$$

where $c_1 = 1/\sqrt{\det(I-K_0)}$.

It's easy to get by induction that the following estimate is true:

$$\begin{aligned}&\left|r_n(t,x,y)\right| \\ &\quad < \int_0^t d\tau \int_{R^n} \left|r_{n-1}(\tau,z,y)M(t-\tau,x,z)\right| dz \\ &\quad < c^{n+1}c_1^n \frac{\pi(n+1)/2}{\Gamma((n+1)/2)} \exp\left\{\frac{(A^{-1/2}K_0A^{-1/2}(x-y),x-y)}{2t}\right\} p_0(t,x,y),\end{aligned}$$

c is a new constant, therefore

$$\begin{aligned}\left|r(t,x,y)\right| &< \sum_{n=0}^{\infty}\left|r_n(t,x,y)\right| \\ &< \frac{c}{\sqrt{t}}\,\mathrm{e}^{ct}\exp\left\{\frac{(A^{-1/2}K_0A^{-1/2}(z-y),x-y)}{2t}\right\}p_0(t,x,y).\end{aligned}$$

Estimating an integral in (2) by means plugging in (6), we get

$$q(t,x,y) < c\left(1+\sqrt{t}\right)\exp\left\{\frac{(A^{-1/2}K_0A^{-1/2}(x-y),x-y)}{2t}\right\}p_0(t,x,y),$$

and the statement of theorem follows. □

Notation 1. In this example the well-known explicit form of solution $p_0(t,x,y)$ of non-disturbed equation was used for:

1) calculating of the function ϕ in relation $p_0(t,z,y)=\phi(t,x,z,y)p_0(t,x,y)$;

2) solving and then estimating the integrals by means substitution (7).

Under disturbing the equation with variable coefficients (on manifold) an explicit form of $p_0(t,x,y)$ is unknown, but the estimate of the function ϕ will be obtained and the analogous substitution (7) will be presented.

2. Scalar perturbation of variable operator

Let $p_0(t,x,y)$ be a fundamental solution (heat kernel) of parabolic equation

$$\frac{\partial u}{\partial t}=\frac{1}{2}\Delta u,$$

where Δ is Laplace–Beltrami operator on complete simply connected Riemann manifold of non-positive curvature M, $\dim M=n$. Denote as $\gamma(s)$ geodezical, parametrized by natural parameter (as a rule, $\gamma(0)=y$, $\gamma(\rho(x,y))=x$), and put $e(x,y)=-\dot{\gamma}(x)$.

The equation on the Riemann manifold is studied in many papers, some of them are [5,6].

Assume, that curvature tensor satisfies the following conditions:

2.1. For arbitrary $x\in M$, $U,V\in T_xM$:

$$\sum_k\left|(R(x)(U,e_k)V,\phi_k)\right| < c\sqrt{\mathrm{Ric}(x)(U,U)\,\mathrm{Ric}(V,V)},$$

where

$$\mathrm{Ric}(x)(U,U)=\sum_k\left(R(x)(U,e_k)V,e_k\right),$$

$\{e_k\}$, $\{\phi_k\}$ are arbitrary orthogonal basises in T_xM, and the constant c doesn't depend on x.

2.2. Along any geodezical the scalar curvature $r(x)=\operatorname{tr}\mathrm{Ric}(x)$ decreases quite fast, that is: $\int_0^\infty sr(\gamma(s))\,ds<c$, where c doesn't depend on γ.

2.3. Co-variant derivative of the curvature tensor satisfies estimates

$$\begin{aligned}
&\big\|(\nabla_{X(s)}R)\big(\gamma(s)\big)\big(Y(s),\dot\gamma(x)\big)Z(s)\big\| \\
&\quad\leqslant f_1\big(\gamma(s)\big)\big\|X(s)\big\|\big\|Y(s)\big\|\big\|Z(s)\big\|; \\
&\big\|(\nabla_{U(s)}\nabla_{X(s)}R)\big(\gamma(s)\big)\big(Y(s),\dot\gamma(x)\big)Z(s)\big\| \\
&\quad\leqslant f_2\big(\gamma(s)\big)\big\|X(s)\big\|\big\|Y(s)\big\|\big\|Z(s)\big\|\big\|U(s)\big\|,
\end{aligned}$$

where f_1 and f_2 are such functions, that along any geodezical γ: $\int_0^\infty s^2 f_k(\gamma(s))\,ds < c$, c doesn't depend on γ.

Let us define along $\gamma(s)$ an operator D on $T_{\gamma(s)}M$

$$D\big(\gamma(s)\big)U = \nabla_U\rho\big(y,\gamma(s)\big)\dot\gamma(x), \quad \dot\gamma(x) = y,$$

and functions

$$\begin{aligned}
&a(x,y) = \mathrm{tr}(D(x) - I), \\
&q(t,x,y) = (2\pi t)^{n/2}\exp\left\{-\frac{\rho^2(x,y)}{2t}\right\}.
\end{aligned} \tag{8}$$

As it was shown in [1–4], under satisfaction conditions 1–3, the following results hold:

1) heat kernel $p_0(t,x,y)$ satisfies two-sided estimate

$$\exp\{-\phi(x,y) - kt\} \leqslant \frac{p_0(t,x,y)}{q(t,x,y)} \leqslant 1,$$

where

$$\phi(x,y) = \frac{1}{2}\int_0^{\rho(x,y)} \big(\rho(x,y) - \tau\big)\,\mathrm{Ric}\big(\gamma(\tau)\big)\big(\dot\gamma(\tau),\dot\gamma(\tau)\big)\,d\tau,$$

k is some constant.

2) the relation holds:

$$\mathrm{grad}_x \ln p_0(t,x,y) = \frac{\rho(x,y)}{t}e(x,y) + w(t,x,y),$$

where $\|w(t,x,y)\| < c$, $x, y \in M$, $0 < t \leqslant T$.

3) the function $a(x,y)$ satisfies the estimate

$$0 \leqslant a(x,y) \leqslant \int_0^{\rho(x,y)} s\,\mathrm{Ric}\big(\gamma(s)\big)\big(\dot\gamma(s),\dot\gamma(s)\big)\,ds,$$

$\mathrm{grad}_x\, a(x,y) < c$, $|\Delta_x(x,y)| < c$.

As a consequence of the relation for logarithmic gradient (condition 2) we have

Lemma 4. *The following inequality is true*:

$$p_0(t,z,y) \leqslant p_0(t,x,y)\phi_1(t,x,y,z)\phi_2(t,x,z),$$

where

$$\phi_1 = \exp\{\rho(x,y)\rho(x,y)\big(e(x,y), e(x,z)\big)\},$$

$$\phi_2 = \exp\left\{-\frac{\rho^2(x,z)}{2t} + c\rho(x,z)\right\}.$$

Proof. Let $\sigma(s)$ be geodezicals, connecting z and x, $\sigma(0) = x$, $\sigma(\rho(x,z)) = z$. Integrating equality (2), we have

$$p_0(t,z,y) = p_0(t,x,y)\left\{\frac{1}{t}\int_0^{\rho(x,z)} \rho\big(y,\sigma(s)\big)\big(e\big(\sigma(s),y\big),\dot\sigma(s)\big)\,ds + \int_0^{\rho(x,z)} \big(w\big(t,\sigma(s)\big),\dot\sigma(s)\big)\,ds\right\}.$$

Notice that

$$\rho\big(y,\sigma(s)\big)\big(e(\sigma(s),y),\dot\sigma(s)\big) = -\frac{1}{2}\frac{d}{ds}\rho^2\big(y,\sigma(s)\big),$$

and, thus, we have an inequality

$$p_0(t,z,y) \leqslant p_0(t,x,y)\exp\left\{\frac{\rho^2(x,y)-\rho^2(z,y)}{2t} + c\rho(x,z)\right\}.$$

We can make stronger, using cosines theorem for manifold of non-positive curvature $\rho^2(y,z) \geqslant \rho^2(x,y) + \rho^2(x,z) - 2\rho(x,z)\rho(x,y)(e(x,y),e(x,z))$, and this leads to desired result. □

Let us consider a perturbation

$$L_1 u = \frac{1}{2}\operatorname{div} b(x)\operatorname{grad} u,$$

where a scalar function $b(x)$ satisfies conditions:

1) $0 < b_1 \leqslant b(x) \leqslant b_2 < 1$;

2) $\|\operatorname{grad} b(x)\| < c$, $\nabla_U \operatorname{grad} b(x) < c\|U\|$.

Put

$$p_1(t,x,y) = \big(2\pi t b(x)\big)^{-n/2}\exp\left\{\frac{\rho^2(x,z)}{2tb(x)}\right\},$$

and define function $m(t,x,y)$ via equality (3). Then error of disturbed equation be

$$M(t,x,y) = \frac{1}{2}\operatorname{div}\big(1+b(x)\big)\operatorname{grad} m(t,x,y) - \frac{\partial}{\partial t}m(t,x,y) = I_1 + I_2, \tag{9}$$

where

$$I_1 = \int_M \left(\frac{1}{2}\operatorname{div}_x b(x)\operatorname{grad}_x p_1(t,x,z) - \frac{\partial p_1(t,x,z)}{\partial t}\right)p_0,$$

and the item I_2 after applying of formula (5) and integrating by the parts has a form

$$I_2 = \int_M \big(\Delta_x p_1(t,x,z) - \|\Lambda(t,x,z)\|^2 p_1(t,x,z)$$
$$- \operatorname{div}_z \Lambda(t,x,z) p_1(t,x,z)\big) p_0(t,z,y)\,\sigma(dz).$$

Lemma 5. *Functions $m(t,x,y)$ and I_1 are bounded by values*

$$c\exp\left\{\frac{b(x)\rho^2(x,y)}{2t}\right\} p_0(t,x,y) \quad and \quad \frac{c}{\sqrt{t}}\exp\left\{\frac{b(x)\rho^2(x,y)}{2t}\right\} p_0(t,x,y)$$

correspondingly.

Proof. From the statement of Lemma 4 (since ϕ_2 is negative) an estimate follows

$$m(t,x,y) < cp(t,x,y)\int_M \big(2\pi t b(x)\big)^{-n/2}$$
$$\times \exp\left\{\frac{\rho(x,y)\rho(x,z)(e(x,y),e(x,z))}{t} - \frac{\rho^2(x,z)}{2tb(x)}\right\}\sigma(dz)$$
$$= c\exp\left\{\frac{b(x)\rho^2(x,y)}{2t}\right\} p(t,x,y)\int_M \big(2\pi t b(z)\big)^{n/2}$$
$$\times \exp\left\{-\frac{\|\rho(x,z)e(x,z) - b(x)\rho(x,y)e(x,y)\|}{2tb(x)}\right\}\sigma(dz).$$

A substitution

$$U = \frac{\rho(x,z)e(x,z) - b(x)\rho(x,y)e(x,y)}{\sqrt{tb(x)}},$$
$$z(U) = \operatorname{Exp}_x\big\{\sqrt{tb(x)}\,U + b(x)\rho(x,y)e(x,y)\big\}$$

transforms the last integral to

$$\int_{T_xM} \left(\frac{b(x)}{b(z(U))}\right)^{n/2} J\big(z(U)\big)\,\mu_x(dU),$$

where μ_x is canonical measure on T_xM, and boundedness of Jacobian J is proved in [6]. Thus, the estimate of $m(t,x,y)$ is obtained. □

For estimating of I_1 we transform the function under integral sign.

$$\frac{\partial p_1}{\partial t}(t,x,z) = \left(-\frac{n}{2t} + \frac{\rho^2(x,z)}{2t^2 b(x)}\right) p_1(t,x,z);$$
$$\operatorname{grad}_x p_1(t,x,z) = \frac{1}{t}\left(\frac{\rho^2(x,z)\operatorname{grad} b(x)}{2b^2(x)} - \frac{\rho^2(x,z)\dot\gamma(\rho(x,y))}{b(x)}\right) p_1(t,x,z);$$
$$(\nabla_U b(x)\operatorname{grad}_x p_1(t,x,z), U)$$
$$= p_1(t,x,z)\left[\frac{1}{t^2 b(x)}\left(\rho^2(x,z)\big(\dot\gamma(\rho),U\big)^2 - \frac{\rho^3(x,z)(\operatorname{grad} b(x),U)(\dot\gamma(\rho),U)}{b(x)}\right.\right.$$

$$+\frac{\rho^4(x,z)}{4b^2(x)}\big(\operatorname{grad} b(x),U\big)^2\bigg)+\frac{1}{t}\bigg(\frac{(\nabla_U\rho^2(x,z)\operatorname{grad} b(x),U)}{2b(x)}$$
$$-\frac{\rho^2(x,z)(\operatorname{grad} b(x),U)^2}{2b^2(x)}-D(x)U\bigg)\bigg].$$

A summation with respect to orthogonal basis $\{e_k\}$ in T_xM $(e_1=\dot\gamma(\rho))$ gives:

$$\frac{1}{2}\operatorname{div} b(x)\operatorname{grad} p_1(t,x,z)-\frac{\partial}{\partial t}p_1(t,x,z)$$
$$=\bigg(\frac{\rho(x,z)(\operatorname{grad} b(x),\dot\gamma)}{2t^2b^2(x)}+s\frac{\rho^4(x,z)\|\operatorname{grad} b(x)\|^2}{8t^2b^3(x)}-\frac{1}{2t}\operatorname{tr}\big(D(x)-I\big)$$
$$+\frac{\rho(x,z)}{2tb(x)}\big(\operatorname{grad} b(x),\dot\gamma(\rho)\big)-\frac{\rho^2(x,z)}{4tb^2(x)}\|\operatorname{grad} b(x)\|^2$$
$$+\frac{\rho(x,z)}{4tb(x)}\Delta b(x)\bigg)p_1(t,x,z).$$

Obtained relation we estimate by the value $\frac{c}{\sqrt{t}}(1+\frac{\rho^4(x,z)}{t^2})p_1(t,x,z)$, and as $(\frac{\rho(x,z)}{\sqrt{t}})^k\varphi_2(t,x,z)$ is bounded, then

$$(I_2)<\frac{c}{\sqrt{t}}\,p_0(t,x,y)\int\limits_M\varphi_1(t,x,y,z)p_1(t,x,z)\,\sigma(dz).$$

The second statement of lemma follows.

We are coming to estimating of the item I_2, containing a logarithmic derivative

$$\Lambda(t,x,z)=-\frac{n}{2}\frac{\operatorname{grad} b(z)}{b(z)}-\frac{\rho(x,z)\dot\gamma(\rho(x,z))}{tb(x)}\in T_xM.$$

Lemma 6. *The following estimate is true*:

$$|I_2|\leqslant\frac{c}{\sqrt{t}}\exp\bigg\{\frac{\rho^2(x,y)b(x)}{2t}\bigg\}p_0(t,x,y).$$

Proof. Note, that

$$\big\|\Lambda(t,x,z)\big\|^2+\operatorname{div}_z\Lambda(t,x,z)$$
$$=\bigg(\frac{n}{2}+\frac{n^2}{4}\bigg)\frac{\|\operatorname{grad} b(x)\|^2}{b^2(x)}+\frac{\rho^2(x,z)}{t^2b^2(x)}$$
$$+\frac{n}{tb(x)b(z)}\rho(x,z)\big(\dot\gamma\big(\rho(x,z),\operatorname{grad} b(z)\big)\big)-\frac{n\Delta b(z)}{2b(z)}-\frac{\operatorname{tr} D(z)}{tb(x)}.$$

The second item under integral sign

$$\Delta_x p_1(t,x,z)=\bigg[\frac{1}{t^2}\bigg(\frac{\rho^4(x,z)\|\operatorname{grad} b(x)\|^2}{4b^4(x)}$$
$$-\frac{\rho^3(x,z)(\operatorname{grad} b(x),\dot\gamma(\rho))}{b^3(x)}+\frac{\rho^2(x,z)}{b^2(x)}\bigg)$$

$$+\frac{1}{t}\left(\frac{2\rho(x,z)(\operatorname{grad} b(x),\dot{\gamma}(\rho))}{b^2(x)}+\frac{\rho^2(x,z)}{2b^2(x)}\Delta b(x)+\right.$$
$$\left.-\frac{\rho^2(x,z)}{b^3(x)}\|\operatorname{grad} b(x)\|^2-\frac{1}{b(x)}\operatorname{tr} D(x)\right)\Big]p_1(t,x,z),$$

and difference

$$\Delta_x p_1(t,x,z)-p_1(t,x,z)\big(\|A(t,x,z)\|^2+\operatorname{div}_z A(t,x,z)\big)$$

is estimated by the value

$$\frac{c}{\sqrt{t}}\left(1+\frac{\rho^4(x,z)}{t^2}\right)p_1(t,x,z),$$

which has integrable with respect to t singularity. Desired statement is proved by the same way like in Lemma 5. □

Corollary 2. *Error $M(t,x,y)$, defined by* (9), *satisfies an estimate*

$$M(t,x,y)<\frac{c}{\sqrt{t}}\exp\left\{\frac{b(x)\rho^2(x,y)}{2t}\right\}p_0(t,x,y).$$

Notation 3. Since the inequalities

$$b(x)\leqslant b_2<1 \quad \text{and} \quad p<q$$

hold, the function $\exp\{\frac{b(x)\rho^2(x,y)}{2t}\}$ is integrable with respect to measure $p_0(t,x,y)\,\sigma(dy)$.

Theorem 2. *Fundamental solution $p(t,x,y)$ of disturbed equation*

$$\frac{\partial u}{\partial t}=\frac{1}{2}\operatorname{div}\big(1+b(x)\big)\operatorname{grad} u$$

satisfies the estimate:

$$p(t,x,y)\ <\ c\exp\left\{\frac{b_2\rho^2(x,y)}{2t}\right\}q(t,x,y)$$
$$\leqslant c\exp\left\{\frac{b_2\rho^2(x,y)}{2t}+\varphi(x,y)+kt\right\}p_0(t,x,y),$$

where functions $q(t,x,y)$ and $\varphi(x,y)$ are determined by formulas (8) *and* (9) *correspondingly.*

Proof. We will prove a convergence of $\sum_{n=0}^{\infty} r_n$ under solution of Volterra's equation, estimating the items of series. From corollary of Lemmas 5 and 6 and the inequality $p\leqslant q$ an estimate follows:

$$|r_1(t,x,y)|=\left|\int_0^t d\tau\int_M M(t-\tau,x,z)M(\tau,z,y)\,\sigma(dz)\right|$$

$$< c^2 \int_0^t \frac{d\tau}{\sqrt{\tau(t-\tau)}} \int_M \big(4\pi^2\tau(t-\tau)\big)^{-n/2} \times \exp\left\{-\frac{(1-b_2)}{2}\left(\frac{\rho^2(y,x)}{\tau} + \frac{\rho^2(x,z)}{t-\tau}\right)\right\} \sigma(dz).$$

Let us transform the inner integral by means substitution

$$u = \left(\sqrt{\frac{t}{\tau(t-\tau)}}\,\rho(x,z)e(x,z) - \sqrt{\frac{t-\tau}{t\tau}}\,\rho(x,y)e(x,y)\right)\sqrt{1-b_2},$$

$$z(u) = \mathrm{Exp}_x\left(\sqrt{\frac{\tau(t-\tau)}{t(1-b_2)}}\,u + \frac{t-\tau}{t}\rho(x,y)e(x,y)\right). \tag{10}$$

The argument of exponent equals here

$$-\frac{1}{2}\|u\|^2 - \frac{\rho^2(x,y)}{2t}(1-b_2) - \frac{(1-b_2)}{\tau}\big(\rho^2(y,z) - \rho^2(x,y) - \rho^2(x,z) + 2\rho(x,y)\rho(x,z)\big(e(x,y),e(x,z)\big)\big),$$

and the relation in brackets is non-negative because of curvature non-positivity. Thus,

$$\int_M (4\pi^2\tau(t-\tau))^{-n/2} \exp\left\{-\frac{(1-b_2)}{2}\left(\frac{\rho^2(y,z)}{\tau} + \frac{\rho^2(x,z)}{t-\tau}\right)\right\}\sigma(dz) \leqslant (1-b_2)^{-n/2} \exp\left\{\frac{b_2\rho^2(x,y)}{2t}\right\} q(t,x,y) \int_{T_xM} J\big(z(u)\big)\,\mu_x(du).$$

So, we have

$$\big|r_1(t,x,y)\big| < c^2 c_1 \pi \exp\left\{\frac{b_2\rho^2(x,y)}{2t}\right\} q(t,x,y),$$

where $c_1 = (1-b_2)^{-n/2}\sup_{z\in M} J(z)$.

It's easy to obtain an estimate

$$\big|r_n(t,x,y)\big| < \frac{c^{n+1}c_1^n \pi^{(n+1)/2} t^{(n-1)/2}}{\Gamma((n+1)/2)} \exp\left\{\frac{b_2\rho^2(x,y)}{2t}\right\} q(t,x,y),$$

providing with absolute convergence of series $\sum_{n=0}^{\infty} r_n(t,x,y)$, $0 < t \leqslant T$ and for the sum of series

$$\big|r(t,x,y)\big| < \frac{c}{\sqrt{t}} \exp\left\{\frac{b_2\rho^2(x,y)}{2t}\right\} q(t,x,y).$$

Then

$$
\begin{aligned}
p(t,x,y) &= m(t,x,y) + \int_0^t d\tau \int_M m(t-\tau,x,z) r(\tau,z,y)\,\sigma(dz) \\
&< c\exp\left\{\frac{b_2\rho^2(x,y)}{2t}\right\} q(t,x,y)\bigl(1+\sqrt{t}\bigr),
\end{aligned}
$$

and this implies, finally, the statement of theorem. □

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